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Geostatistics 1

Lognormal-de Wijsian Geostatistics for Ore Evaluation

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Preface

In 1962 Dr Danie Krige published a monograph on 'Statistical Applications in Mine Valuation' under the aegis of the Institute of Mine Surveyors of South Africa. This publication has given yeoman service as a standard text for the teaching of mining engineers at the University of the Witwatersrand, and for others interested or involved in estimating the ore reserves of a mine or mining prospect. The present monograph is a worthy successor which updates further analytical work by the author, and correlates it with the theoretical concepts developed by Professor G. Matheron of the Centre de Morphologie Mathématique of Fontainbleau, France.

The significance of this work is primarily due to the fact that a mine is very seldom so fortunately placed as to be able profitably to mine all the mineralized ore comprising an ore body. Normally, the distribution of mineralization is such that the average grade of portions of the ore body is too low to support profitable mining. In these circumstances some form of selective mining must be practised. This requirement focuses attention on two fundamental problems that can only be solved effectively by the application of statistical methods of analysis, namely:

● the need to estimate the 'global percentage payability' of the ore body and the average grade of the payable ore, that is, to estimate the average grade and total tonnage of the ore reserves;

● the need to identify the average grade and tonnage of individual mining blocks for the purpose of planning selective mining and grade control.

Krige originally developed solutions for both these problems by clear-sighted statistical analysis of a carefully assembled mass of data derived from continuous sampling records on the South African gold mines over a long period of time. These pragmatic procedures were subsequently vindicated by Professor G. Matheron who evolved the general statistical theory of geostatistics applicable to any regionally correlated variable. What higher acknowledgement could anyone be paid than to have his name perpetuated in the technical literature of his chosen field? This honour Matheron generously bestowed on Dr Krige when he gave the appellation *kriging* to the mathematical procedure for calculating the best linear unbiased estimator of the grade of an ore block as the weighted mean of surrounding sample values (what Krige originally called the weighted moving average).

The theory developed in this monograph is based essentially on the application of a de Wijsian spatial model to a log normally distributed variable. It is of interest to note how closely this model approximates to the observed behaviour of gold values in the South African gold mines, as demonstrated by the numerous practical examples included in the text. In addition, Krige indicates the potential for further extensions of this approach to other ore bodies and other parameters, for example, the analysis of the undulating foot-wall and hanging-wall mining limits for the sulphide ore body of the Prieska copper mine.

The application of geostatistics has now been established on a worldwide basis as a valid method for deriving a statistically sound estimate of mine ore reserves. Development is continuously taking place as a result of the efforts of numerous practitioners and theoreticians. However, the fundamental principles established by Krige and Matheron have withstood the test of time. This monograph presents an introduction to the subject in the typical clear thinking manner of the 'maestro' who has been actively involved in its development from its inception.

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Introduction

This monograph is based on the notes prepared for a workshop course for the Australian Mineral Foundation, held in Adelaide in June 1977. It is intended to provide a brief review of the developments in the application of geostatistics in South African ore valuation, mainly on the gold mines. The applications covered include the earlier statistical developments which preceded and led to the establishment of the more 'spatially' orientated statistics now generally known as geostatistics. The detailed development of all the relevant models and formulae used is not covered; should the reader desire a thorough knowledge of these models, this monograph should be studied together with the listed references.

The metric system has been used throughout except for most of the historical examples of, and references to, earlier developments in the field of geostatistics predating the introduction of metrication in the local mining industry. Metrication of such examples and references could obscure the other-wise obvious choices of, for example, value or distance intervals, and would in any case not affect the principles involved; also, for practical reasons, the unit areas used in geostatistical ore reserve valuations on some of the gold mines since the early 1960s have not yet been, and are unlikely to be, metricated.

This is a revised version of the original monograph published in 1978 and incorporates a number of corrections and improvements.

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D. G. KRIGE
Johannesburg, 1981

SECTION 1*

Statistical Concepts, Procedures and Applications

(Main references: Krige, 1951a, 1951b, 1960, 1962, 1966a; Storrar, 1977)

1.1 Statistics and probability

Statistics is essentially a study of variability, the use of some suitable mathematical model representative of such variability, and the application of this inferred pattern of behaviour to practical problems. Probability theory is the mathematical structure devised for providing models for chance happenings. Perhaps the simplest example of a chance happening is the result obtained in tossing a coin. Although the result of a single toss can never be predicted with certainty, there is no reason why in repeated tosses the expectancy of heads and tails should not be equal. In such a case the outcomes of the experiment are said to be equally likely, so that the probability of obtaining heads (or tails) is 1 in 2, that is, 0,5. This simple probability model refers to a 'mathematical' or 'unbiased' coin. Whether such probabilities are meaningful for actual coins tossed in any particular way is a matter for experience to determine.

Probability models have also provided the basis for representing observed variability in cases where the actual underlying probabilities are only partly known or defy analysis. A typical example of the latter case is the variability observed in the values obtained from gold or other ore samples. The interacting natural forces which gave rise to this variability are obviously very complex and are unlikely ever to be unravelled completely. In spite of this, probability theory and the more recent geostatistical theories have provided us with models which can represent this variability on a reasonable basis. Applied statistics and geostatistics must therefore never be accepted as exact mathematical sciences. Furthermore, a basic and essential concept is that in the type of applications covered in these notes, no model must ever be accepted as necessarily the final and exact answer to the problem; *thus, any model must prove and continue to prove its appropriateness in practice, preferably by follow-up comparisons.*

1.2 Patterns of variation

It is expected that the reader is familiar with basic statistical concepts and graphical representations such as frequency (observed, percentage, cumulative), value categories, category limits, histograms, frequency distributions, and frequency distribution curves. Figure 1 shows the observed histogram and frequency distribution curve for 28 334 gold values from a section of the Blyvooruitzicht gold mine (Krige, 1951b). The skew bell-shaped curve is typical of gold value and many other ore value distributions and is an excellent example of the *lognormal curve*.

1.3 Brief historical review

The recognition of a pattern of distribution for gold values dates back some 60 years. It started with Hooper, then Chief Surveyor on the West Rand Consolidated gold mine, who collected large numbers of underground sample values and found that these, when segregated in value categories, disclosed a distinctive frequency pattern. Watermeyer, of the University of the Witwatersrand, cooperated with Hooper and, in January 1919, published the first paper on this work (Watermeyer, 1919). Two aspects of this original work are interesting from a statistical point of view. Firstly, Watermeyer fitted frequency curves to the positively skewed histograms on the basis of two halves of two different normal curves joined at their modes. Needless to say, the fits were not particularly successful. Secondly, he recommended a basis for arriving at a reliable mean value for a limited number of gold values, which in essence meant the weighting of values by (frequency)^2 instead of by frequency only, as for the conventional arithmetic mean. His main reason for discarding the arithmetic mean was that the latter was reliable and unbiased only when it coincided with both the modal and median values; that is, where the distribution was symmetrical about the mean value, a requirement not satisfied in the case of gold values.

A decade later, Truscott (1929) revived the interest in this form of approach by suggesting that values should be weighted not only by (frequency)^2, as proposed by Watermeyer, but in addition by the values themselves, so as to allow for the distribution of values laterally away from an exposed face as distinct from the distribution observed only longitudinally on the face. These early crude attempts at the use of statistical techniques failed on a wrong interpretation of two basic principles, namely:

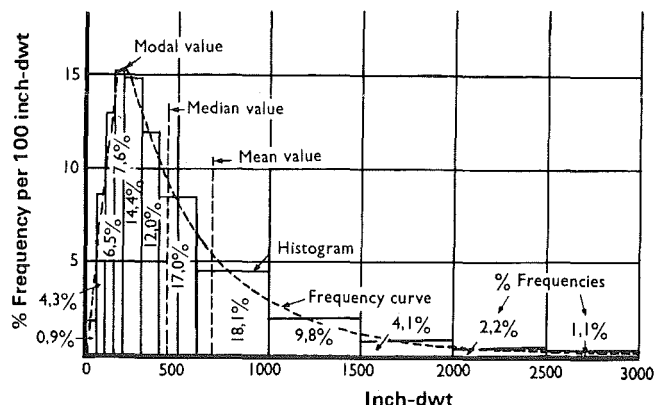(i)   that the arithmetic mean of a set of values drawn at

Fig. 1.   Histogram and frequency curve for 28 334 gold values from the Blyvooruitzicht mine.

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random from an unknown population always provides an unbiased estimate of the population mean no matter what the shape of the frequency distribution of the population of such values; and

(ii) that where such a set of values is not obtained in a random manner but systematically, and then only on the periphery of an ore block such as in our gold mines, there is still no valid reason why the arithmetic means of such sets should be consistently too high for all the ore blocks throughout the whole mine.

It is only fair to mention that several contributors to both these papers raised valid queries against the discarding of the arithmetic mean and that neither of the two suggested procedures was ever generally accepted in practice. A further interesting point was that, although the distribution pattern was not recognized as approximating to the lognormal until Sichel started his research work at the Rand Leases mine in the forties (Sichel, 1947), Truscott came very close to it when he observed, in 1930, that a histogram of gold values, if drawn to a logarithmic base, disclosed symmetry around the ordinate of the geometric mean.

Credit for the first concrete applications of statistical procedures must go to Sichel for his work on bias errors introduced in the manual taking of gold ore samples underground, which led to his paper published in 1947. He proposed the use of the lognormal model, and illustrated the serious extent of possible bias errors in ore sampling, particularly on narrow reefs. This work was followed up shortly afterwards by Ross (1950) and the author (1951a), and since then the field of application for statistical techniques in mine valuation has expanded rapidly.

In the early 1950s De Wijs also started his research along somewhat different lines (De Wijs, 1951). The title and contents of this monograph however, provides evidence of how his and the local approaches, as well as the later French research (Matheron, 1960) merged into a coherent whole.

The main concepts underlying the application of statistics and geostatistics to ore valuation are based on common sense principles as old as the science of ore valuation itself. The fact that an exceptional value in a small group of ore values is recognised as such implies that the other values follow some pattern and that the 'outlier' does not fit this pattern. *The basic statistical concept* of such a 'pattern' is that of a frequency distribution model. Similarly, common sense has also dictated that where two boreholes, say, have been drilled far apart on a property and one has been deflected twice, the four available intersections cannot be given equal weight in an overall grade evaluation. This implies the *basic geostatistical concepts* that grades close together in an ore body tend to be more similar and that an appropriate weighting system which takes account of the relative locations of the corresponding samples is called for.

1.4 Populations and units of measurement

In statistics the set of observations is conveniently interpreted as a *sample* drawn from a *population* of all conceivable observations that might have been made of the same type. In many cases the collection of objects to which the observations relate can readily be visualised as a population, for example, the inhabitants of a city, a collection of chips in a bowl, or a box of parts produced by a specific manufacturing process. It is the numerical characteristic observed in respect of these objects (for example, the height of individuals) which forms the *variable* to be studied, and the collection of all possible values of this characteristic designates the population. In the case of the ore values referred to above, the objects concerned were the 28 334 ore samples (Fig. 2) taken at 5-foot

(1,52 m) intervals along the drives and raises in the relevant section of the mine; the characteristic studied was the inch-dwt (strictly, inch-dwt/t) values of these ore samples (1 inch-dwt = 4,3543 cm g/t). These observed inch-dwt values, replaced by cm g/t values since metrication, constitute a *sample* (in the statistical sense) drawn from a *population* of all such possible values which could be obtained by repeated *sampling* (in the mining sense) of the ore exposed in that section of the mine. For all practical purposes the population concerned can be accepted as *infinite* as distinct from *finite* populations such as are represented by, for example, the collection of chips in a bowl.

More than one characteristic of the same collection of objects may require study — thus, apart from the gold cm g/t values of the ore samples, their uranium inch-lb (now cm kg/t) values or their pyrite inch-percent (now cm %) values, or their weights, or their grading analyses may also require investigation. This is particularly the case where it is suspected that such characteristics may be related to one another in some way, that is, that they may be *correlated*. In such cases each set of values will constitute a statistical sample drawn from a population of values distinct from the population of gold cm g/t values, but not necessarily uncorrelated with them.

We can now retrace our steps to the concept of probability and interpret the probability of an event as the relative frequency with which the event is observed to occur in the population. Where the complete population is not known and can never be observed, as in the case of inch-dwt values referred to above, deductions are made from the available sample of values, the pattern (or frequency distribution) of which must obviously approach that of the population more closely the larger the sample of values is made. In the above example, for instance, the sample is so large that the probability of, for example, the percentage of values to be obtained in the category between 99,5 and 149,5 inch-dwt being 6,5 % on repeated sampling of the area, must be subject to very small limits of error. Expressed differently, the step-diagram is based on such a large number of values that the smooth curve drawn through the steps must represent closely the actual frequency distribution curve of the population.

It is essential that the basic unit or member of the population be defined clearly. In the valuation of gold mines the inch-dwt or cm g/t values represent direct measurements of *the quantity of gold per unit area* of reef horizon, 36,6 cm g/t being equivalent to a concentration of one gram of gold per square metre of reef:

1 cubic metre of rock *in situ* weighs 2,73 metric tons (density $\rho = 2,73$ t/m^3).

$$\text{Gold content/m}^2 = \{2,73 \times \text{g/t (over stope width)}$$
$$\times \text{ stope width (cm)}\} /100$$
$$= \text{stope width} \times \text{g/t} \times 0,0273 \text{ g/m}^2.$$

But stope width $\times$ g/t $=$ cm g/t value, therefore
gold content (in g/m^2) $= 0,0273$ cm g/t,
and hence 36,6 cm g/t $= 1$ g/m^2.
Similarly, 80 inch-dwt $= 1$ oz per sq fm (1 fm$^2 = 36$ sq ft).

The unit cm g/t developed naturally from the fact that, in practice, assays are reported in g/t; the cm g/t corresponds to the product of this assay value and the width over which the sample was taken from the ore body.

This is unfortunate because it has led or misled many to believe that, for essentially two-dimensional ore bodies, the cm g/t is not a basic unit but a combination of the two *real* basic units of g/t and width. In fact the *cm g/t and width are the two basic variables.* Furthermore, representative values of these two units can be averaged directly to give meaningful

averages over the area from which they were drawn, whereas g/t needs to be weighted with width first. This aspect can be of critical importance when applying statistical and geo-statistical models.

In geostatistics the cm g/t and similar measures are referred to as *accumulations*. It should be stressed that in ore bodies *where the density varies significantly* across the width and/or within the plane of the ore body, *and particularly where it is correlated with ore grade*, the correct unit for accumulations will be $\rho \times$ cm g/t (or $\rho \times$ cm%) and density should then be incorporated as a weighting factor in all estimating procedures. The same will apply in the case of a massive three-dimensional ore body where the correct accumulation unit will be metal content per unit volume equivalent to density $\times$ metal content per unit mass, for example, $\rho \times$ g/t.

The development inch-dwt or cm g/t values dealt with above, therefore, represent the distribution of the measured gold concentrations per unit area for those reef areas which are extracted in cutting the sample channels across the ore body (areas of some 6 sq in or 40 cm² each) in the relevant section of the mine. Similarly, if, for example, the cm g/t values of ore blocks in that section are collected, the unit of this population becomes the relative concentration of gold in an ore block. Also, in the case of single borehole values from a mining property, the virtually infinite number of borehole cm g/t values which could be obtained by repeated drilling would represent the population of relative gold concentrations in reef areas of approximately 7 cm² each. *From the above it is clear that not only the nature of the members of the population should be clearly defined, but also the physical limits of the population itself.*

It should be noted that the gold recovered from a mine during a specific period can be expressed directly as a cm g/t or inch-dwt value by, in effect, spreading the gold over the total area of reef extracted in stoping and development during the same period:

$$\frac{\text{gold recovery}}{\text{(in cm g/t)}} = \frac{\text{gold recovered (in kg)} \times 36\,600}{\text{square metres extracted}},$$

$$\frac{\text{or gold recovery}}{\text{(in inch-dwt)}} = \frac{\text{gold recovered (in oz)} \times 80}{\text{fathoms extracted}}.$$
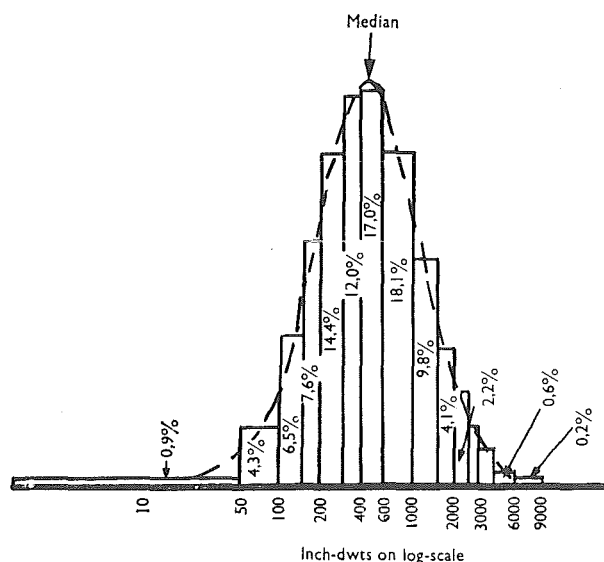
Fig. 2. Normal histogram and frequency curve on a log-scale for 28 334 gold values from the Blyvooruitzicht mine.

Naturally the same principles will apply to any mineral in a deposit which can be accepted as a two-dimensional ore body. The distinction between actual and observed gold concentrations should also be noted. In valuation the actual cm g/t values are never known as the sampling and assay procedures are not perfect and, therefore, we are always dealing with imperfect measurements of such values. A population of ore values, as observed, therefore always arises from a basic population of actual values where the members of the latter population have all been changed in value due to imperfect observation.

1.5 Sampling from populations

Sampling (in a statistical sense), that is, the selection of part of a population to represent the whole population, is a long-established practice. Where the population consists of units which are well-mixed and do not differ radically from one another — for example, grains of corn in a sack — any sample, even if small, is likely to be closely *representative* of the whole. When, however, the units are very dissimilar and are not well-mixed such as in underground gold values, a small sample of these units may, in essential characteristics, differ radically from the population. The method of sampling may also contribute to the extent to which a sample may be unrepresentative mainly in the sense of introducing a *bias*; for example, in sampling a heap of coal round the edges only, the sample will contain too great a proportion of large lumps and, therefore, will be biased. This latter type of error cannot be eliminated by taking larger and larger samples and, therefore, is particularly dangerous. Where bias is eliminated, there will, however, still be the inevitable errors of sampling arising from the chance differences between members of the population included in the sample and those not included. The magnitude of these so-called *sampling errors* will depend on the size of the sample taken, the variability of the material, the structure of this variability, the sampling procedure, and on the method of calculating the results. A *random sample* is obtained when the drawing of every unit of the sample is performed in such a way that each unit in the population has the same chance of being drawn. A *stratified random sample* will result from the division of the population into strata (units within a stratum being as similar as possible), and then sampling each stratum at random. For example, in sampling the population of a large town for income per head, the various suburbs will represent natural strata of the population; if each suburb is, therefore, sampled at random the combined sample for the town will constitute a stratified random sample.

In ore valuation on the South African gold mines (as for bedded deposits of most minerals), ore samples are taken in development and along stope faces at regular intervals. The ore in between sampling sections as well as the ore inside the relevant ore blocks, therefore has no chance of being selected, and the values obtained cannot be regarded as random samples from the populations of values in the ore blocks. At the same time we know from experience and intuition that gold values become relatively less variable the smaller the reef area containing such values becomes; thus, the reef body can also be divided into natural strata. The sampling carried out in valuing an ore block is, therefore, strictly *systematic perimeter sampling* or approximately *stratified random perimeter sampling*. In practice underground, the available samples represent a very irregular pattern of positional distribution and assume a systematic character only over very limited areas.

In the case of borehole drilling from the surface, the positioning of the boreholes should ideally be on a regular

systematic grid but this is never possible for various practical reasons, and the pattern is again usually irregular.

The standard statistical procedures discussed in the major part of Section 1 are based on *random sampling theory* and are, therefore, strictly speaking not applicable to most ore valuations, but can nevertheless be used with the necessary reservations in certain applications; they also form a necessary and useful foundation for the geostatistical procedures covered in Sections 2 and 3.

$$= m = \frac{1}{n} \sum_{i=1}^{i=n} x_i,$$

where x_i comprises all the members of the population, i.e. $n \to \infty$.

Median = middle value when values are listed in ascending or descending order, and corresponds to the value at the 50% cumulative frequency.

Mode = value corresponding to highest frequency per unit interval of value, that is, the peak of a unimodal frequency distribution curve.

1.6 Main distribution parameters

1.6.1 *Measures of location*

Mean = arithmetic mean of the population, corresponding to the centre of gravity of the frequency distribution,

1.6.2 *Measures of variability (or spread)*

Variance = mean of squares of deviations of individual values from their mean (analogous to the moment of inertia),

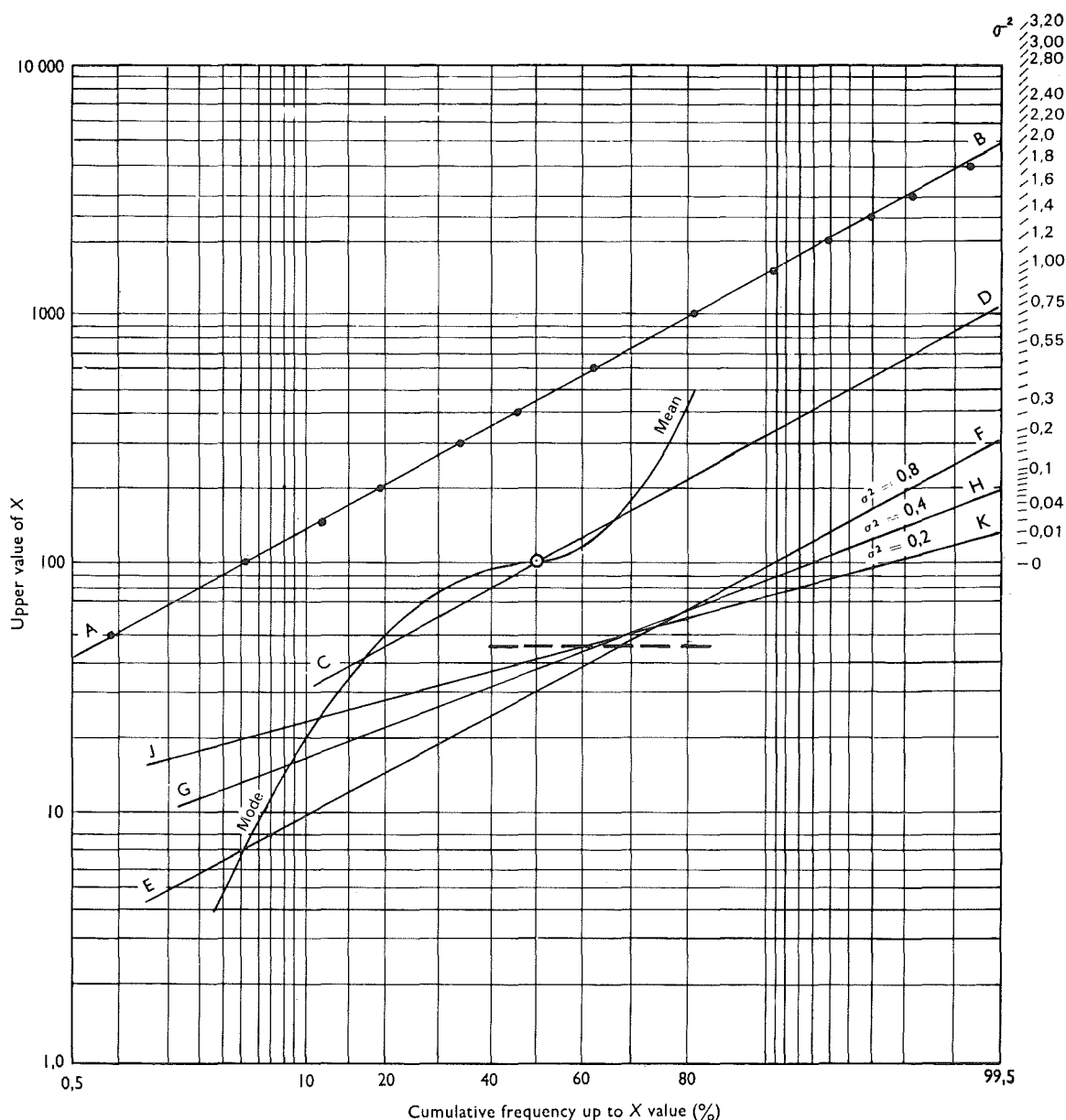$$= \sigma^2 = \frac{1}{n} \sum (x_i - m)^2$$

Fig. 3. Lognormal distributions plotted as straight lines on logarithmic probability paper.

$$= \frac{1}{n} \sum x_i^2 - m^2 \text{ for a population.}$$

For a sample, with $\bar{x}$ = mean of x_i values,

$$Variance \quad = S^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2$$

$$= \frac{1}{n-1} \left[\sum x_i^2 - \bar{x} \sum x_i \right]$$

$$Standard\ deviation = \sqrt{\text{variance}}$$
$$= \sigma \text{ for a population, or}$$
$$= S \text{ for a sample.}$$

In the case of symmetrical distributions the mean, median and mode will coincide. For a skew distribution these three measures will not coincide and their relative positions will depend on the skewness of the distribution.

1.7 The normal frequency distribution

The main features of this distribution are:

Limit of deviations either side of mean	% Frequency between these limits
$\pm\ 0,674\ \sigma$ ('probable error')	50 %
$\pm\ 1,645\ \sigma$	90 %
$\pm\quad 2\sigma$	95 %
$\pm\quad 3\sigma$	99,8 %

The means of samples of size n drawn at random from a normally distributed population, will also be normally distributed with

$$Variance = \sigma_s^2 = (\text{population variance})/n.$$

1.8 The lognormal frequency distribution

As illustrated in Fig. 1 for the Blyvooruitzicht mine, gold values on the South African gold mines are not distributed symmetrically on either side of the mean value, but are positively skew to the right with the mode to the left of the mean and with a long drawn-out tail towards the higher values on the right. Such distributions start at zero and have mode, median and mean values in this sequence. If the histograms of such values are now plotted on a logarithmic scale, the distribution pattern changes radically and approximates to the normal distribution — hence the term *lognormal distribution*. Figure 2 shows the equivalent of Fig. 1 on this basis with a normal curve fitted to the histogram of actual observations. For the purpose of plotting a lognormal distribution as a straight line, *logarithmic-probability paper* is used (Krige, 1951a, 1962). Figure 3 shows the Blyvooruitzicht data plotted as a cumulative frequency distribution on such paper (points plotted along line AB) and it is evident that the lognormal model is suitable for representing this data.

Considering now the normal distribution of the logarithms of the values as shown on Fig. 2, it becomes evident that the mean of this distribution coincides with its mode and median values and at the same time corresponds to the geometric mean of the original untransformed data, because the log of the geometric mean of a set of values equals the mean of the logs of those values. On Fig. 1, therefore, the median value will also coincide with the geometric mean of the population. Furthermore, it can be shown mathematically that for a lognormal distribution there is a definite relationship between the geometric mean and the mean (as well as the mode) dependent only on the variance of the corresponding normal distribution of the logarithms of the values (Krige, 1951a).

Thus

	mean	= (geometric mean) $\times\ e^{0,5\sigma^2}$,	(1)

and

	mode	= (geometric mean)$/e^{\sigma^2}$,	(2)

where σ^2 = variance of Naperian logarithms of values,
= 5,3019 $\times$ variance of logs of values to the base 10.

Also, population variance = mean^2 $(e^{\sigma^2} - 1)$.   (3)

These relationships are best illustrated graphically on Fig. 3 for the Blyvooruitzicht data. The slope of the straight line AB fitted to the data on log-probability paper (see also Rendu, 1978; Storrar, 1977) provides an estimate of the variance, in this case in respect of the logs of the values (to the base e), that is, of σ^2 above. If a parallel line CD is drawn through the centre point of the graph paper,

(i) the intersection of such a line on the σ^2 scale at the extreme right-hand side provides an estimate of σ^2 ($= 0,84$);

(ii) the intersection of CD with the curve marked 'Mean' provides (on the vertical value scale) the value of the factor $e^{0,5\sigma^2} \times 100$ ($= 153$), and also (on the horizontal scale) the cumulative frequency ($= 67,7\ \%$) at which the mean value occurs; and

(iii) the intersection with the curve marked 'Mode' provides (on the vertical value scale) the value of $(1/e^{\sigma^2} \times 100)$ ($= 42,4$) and (on the horizontal scale) the cumulative frequency at which the modal value occurs ($= 17,7\ \%$).

The median value on line AB at the 50 % cumulative frequency, the geometric mean, is 445 inch-dwt and the graphical estimates are therefore:

Mean   = (153/100) $\times$ 445 = 681 inch-dwt, or
= 679 inch-dwt on line AB at 67,7 % cumulative frequency.

Mode   = (42,4/100) $\times$ 445 = 189 inch-dwt, or
= 189 inch-dwt on line AB at 17,7 % cumulative frequency.

As a further graphical aid to the lognormal distribution, Fig. 4 (Krige, 1962), based on (6) and (7), (§1.9), shows for any unit of measurement the relationships between:

(i) the 'pay value/mean value', that is, the average value above any specified (pay limit) value measured in terms of units of mean value (horizontal scale);

(ii) the 'pay limit/mean value', that is, the specified value under (i) in terms of units of mean value (vertical scale);

(iii) the σ^2 value in the range 0,01 to 3,0; and

(iv) the percentage of the values in the distribution above the value specified in (i) above.

1.8.1 *Examples illustrating use of Fig. 4*

In a lognormal distribution with $\sigma^2 = 0,5$, 75 % of the values will lie above a value equivalent to 0,484 times the mean value of the distribution, and these values (75 %) will have a mean value equivalent to 1,221 times the mean value of the distribution (point A). Also, for the Blyvooruitzicht data with $\sigma^2 = 0,84$, the 50 % cumulative frequency (the geometric mean) value should equal 0,66 times the mean value (point B).

Thus, 445 (as observed on Fig. 3) should equal 0,66 $\times$ 681 = 449 inch-dwt.

Figures 3 and 4 are also useful to illustrate the different types of distributions obtained when not dealing with individual underground values. The lines EF, GH and JK on Fig. 3 represent the distributions of individual sample values, means of values in 15-m development stretches, and block values on a gold mine, with σ^2 values of 0,8, 0,4 and 0,2 respectively. The overall mean value is 455 cm g/t in each
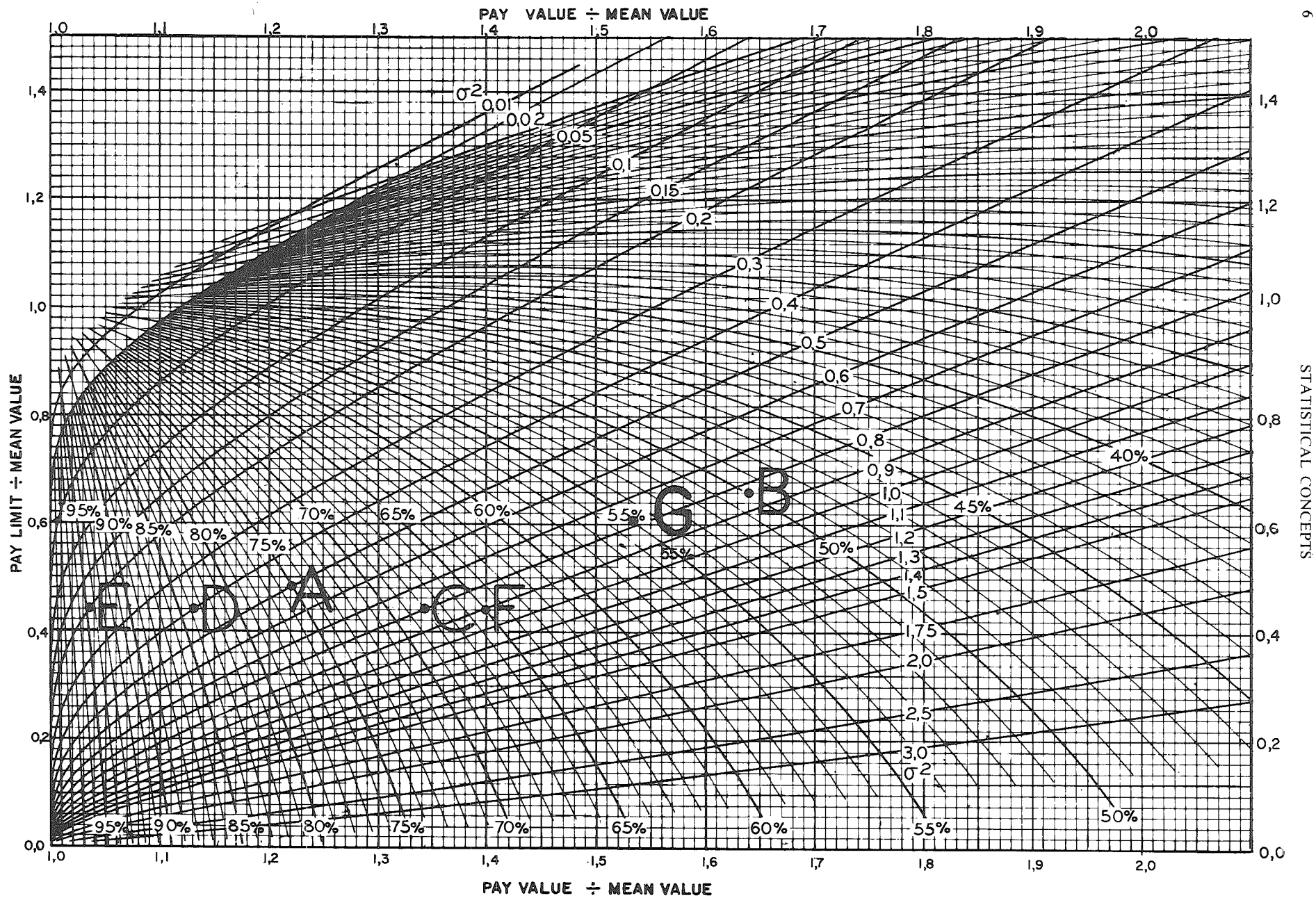
Fig. 4. Graphical relationships between mean value, pay limit, pay value and percentage pay for the lognormal distribution.

case and a theoretical pay limit is accepted at 200 cm g/t; that is, 0,44 times the mean value. (These distributions have been plotted in terms of 1/10th of the actual values, but in the case of lognormal distributions this does not affect the basic characteristics.) The intersections of these lines with the pay limit value line of 200 (20 on the scale chosen) correspond to cumulative frequencies up to the pay limit of 32 %, 16½ % and 5½ % respectively, indicating that:

(i) the development results, if published on the basis of individual sampling sections, would show a payability of 68 %,

(ii) and if published on 15-m stretch values will show 83½ % payability, compared with

(iii) an actual payability, if selective mining is based on ore blocks, of 94½ %.

Referring to Fig. 4, these three payabilities can also be read off for a pay limit of 0,44 (times the mean value) and for σ^2 values of 0,8, 0,4 and 0,2 (points C, D and E), and in addition the corresponding average pay values are read off on the horizontal scale at 1,34 (609 cm g/t), 1,132 (515 cm g/t), and 1,035 (471 cm g/t), respectively.

This illustrates how *payable borehole values*, which are analogous to the payable individual development sampling sections (line *EF* in Fig. 3) will, on a straight percentage basis, understate the percentage of payable ore blocks (line *JK*) on a mine (68 % instead of 94½ %) and overstate the average grade of these blocks (609 instead of 471 cm g/t).

In reverse, Fig. 4 can also be used to estimate σ^2 for the distribution of development values and percentage payability as published. A gold mine published for the year ended 30 December 1960 a mean value in development of 400 inch-dwt of which 65 % was reported as payable at an average value of 560 inch-dwt (that is, 1,4 × mean value). From the diagram (point *F*) these figures indicate that the pay limit was of the order of 0,44 × mean value (= 176 inch-dwt), and that the individual sample values (on the basis of which the development payability was published) were distributed with a logarithmic variance of about 0,9.

1.9 The three-parameter lognormal distribution

An investigation (Krige, 1960) of a large number of gold (and uranium) value distributions from a cross-section of large gold mines showed that these distributions generally do not follow the two-parameter lognormal form but display a regular pattern of departure from it. On the log-scale the histograms all showed a varying degree of negative skewness, with modes to the right of the means. By adopting a transformation to log (value + a constant) instead of only log (value), this skewness can be eliminated; thus, the logs of the values, after first having added a constant — which is the *third parameter* — can be represented by a normal distribution. Although subject to certain theoretical limitations, tests have indicated the practical utility of this modified lognormal model over the full range of conditions likely to be encountered in large South African gold mines and it recom-
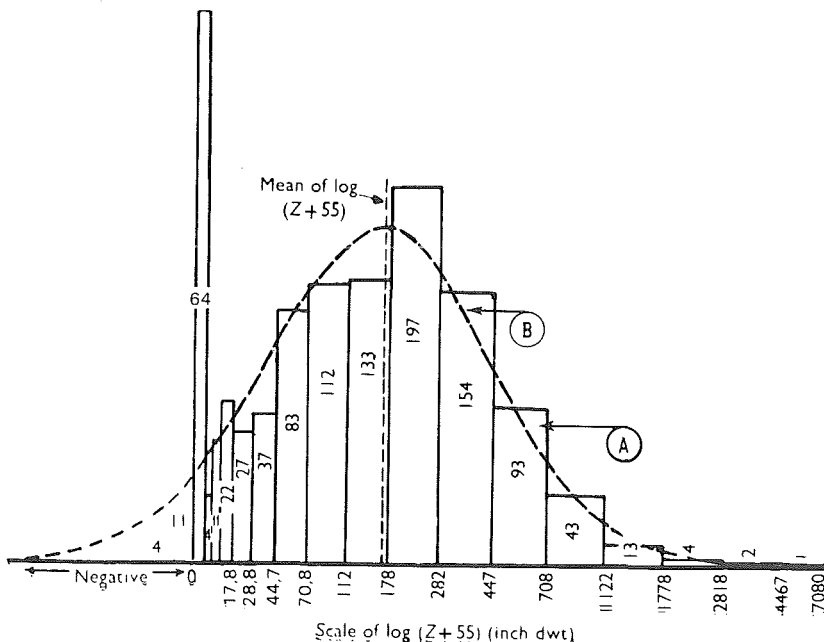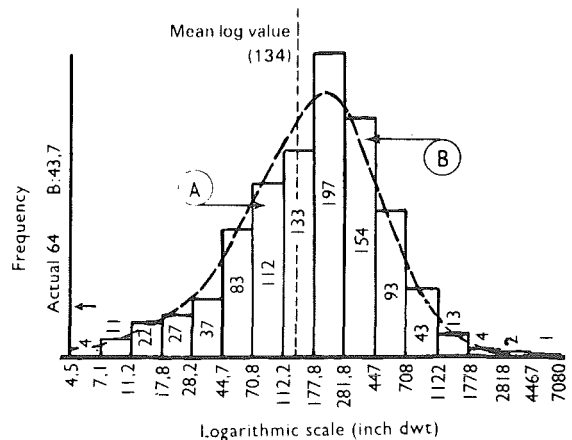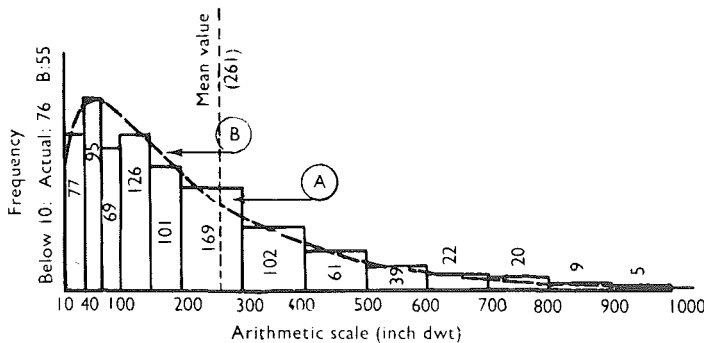
Fig. 5. (*above left*) Histogram of 1000 gold values from the Merriespruit mine.
Fig. 6. (*above right*) As Fig. 5 but with results plotted on a logarithmic scale.
Fig. 7. (*left*) As Fig. 6 with logarithmic scale for (values+constant).

mends itself as a relatively simple and effective model. It has also proved suitable for most base metal distributions encountered on the Prieska copper mine.

An example of the frequency distribution pattern concerned is depicted in Figs 5–8 for 1000 individual inch-dwt values as selected in 1960 on a regular grid from the underground development values at individual sampling sections for the No. 1 and No. 2 shaft workings of the Merriespruit mine, now part of the Harmony mine in the Orange Free State. This case illustrates one of the more extreme departures from the straight lognormal model.

Figure 5 shows the histogram (marked A) on an arithmetic scale for the inch-dwt value categories, whereas Figs 6 and 7 are based on arithmetic scales for log (inch-dwt values) and log (inch-dwt values + 55). The basic distribution pattern (A in Fig. 5) is extremely skew on the positive side with the main peak (or mode) well to the left of the mean value and with a second peak (not evident) in the category below 10 inch-dwt. The distribution of the logarithms of the inch-dwt values (A in Fig. 6) is negatively skew, the mode lying to the right of the mean log value. On the basis of log (value + 55), however (A in Fig. 7), the distribution can be regarded as symmetrical and is reasonably well represented by the Normal (Gaussian) curve of error shown on the figure by curve B. The corresponding transformations of this theoretical curve are also shown on Figs 5 and 6 by curves B.

Figure 8 is a representation on logarithmic-probability paper, A being the cumulative frequency curve for the actual inch-dwt values. The corresponding points for (inch-dwt values + 55) are shown by the filled-in circles, and again the agreement is close between these and the theoretical straight line B, which represents the equivalent of the normal curve for log (value + 55) on this diagram. This graph is convenient for estimating the value of the constant (55), by adding ever-increasing constants to the basic values of the distribution, until the plot of the points best resembles a straight line (see also Rendu, 1978).

The detailed analyses from some 24 large mines (Krige, 1960) have confirmed that this basic pattern persists throughout (for gold, uranium and pyrite), with different values for the additive constant being required. In the case of Blyvooruitzicht shown on Fig. 1, for example, this constant for gold is zero. Although the differences between the actual distributions and the corresponding theoretical curves could not in all cases pass the customary strict statistical X^2 tests (in many cases because of the very large frequencies involved), the effective position is that in practical applications the theoretical model nevertheless proved suitable and sufficiently accurate for all practical borehole valuation purposes for which it is to be used (Krige, 1960).

A distinct advantage of the model based on log (value + a constant) is that estimates of the population mean value framed on it are evidently not very sensitive to variations in the value of the constant ranging from just below to well in excess of the optimum figure. Therefore it could be practical, for example, to accept this constant for gold values at, say, 260 cm g/t for the Basal Reef in the Orange Free State field, and at, say, 170 cm g/t for both the Vaal Reef in the Klerksdorp field and for the Kimberley Reef in the Kinross field. This model has also been found suitable for a variety of other metals such as platinum and base metals.

The following key features of the three-parameter lognormal distribution are worth recording (Krige, 1951b):

(i) Distribution function for lognormal variable z:
$$\psi(x) = [\sigma\sqrt{(2\pi)}]^{-1} . \exp[-(1/2\sigma^2)(x-\xi)^2], \quad (4)$$
where x = normalised version of variable z,
$$= \log_e(z + a),$$
$$\xi = \text{mean of } \log_e(z+a),$$

a = additive constant (3rd parameter), and
σ^2 = variance of $\log_e(z + a)$.

(ii) Mean of z population: $m = \exp[\xi + \tfrac{1}{2}\sigma^2] - a.$ (5)

(iii) The frequency of $(z + a)$ values above a pay limit value of $(z_i + a)$, and hence of z values above z_i,

$$= \frac{1}{\sqrt{(2\pi)}} \int_{w_i}^{\infty} \exp(-\tfrac{1}{2} w^2)\, dw, \quad (6)$$

where $w_i = \dfrac{1}{\sigma}\left[\log_e(z_i + a) - \xi\right]$

and the integral is obtained from tables of the normal distribution.

(iv) The average of all z values above z_i

$$= (m + a)\,\frac{\displaystyle\int_{w_i-\sigma}^{\infty} \exp(-\tfrac{1}{2} w^2)\, dw}{\displaystyle\int_{w_i}^{\infty} \exp(-\tfrac{1}{2} w^2)\, dw} - a. \quad (7)$$

Figure 4 is based on a series of solutions for (6) and (7).

1.10 Sampling from lognormal populations

The arithmetic mean of a sample from a lognormal distribution (as for any type of distribution) will be an unbiased estimate of the population mean, if the population is regarded as an infinite number of individual values obtainable on repeated sampling following the same procedures. However, the physical sampling can introduce a bias in relation to the population of actual values, as is almost invariably the case in underground sampling. This is generally accepted as the main reason for a Mine Call Factor – the ratio of gold accounted for on the surface to gold called for from
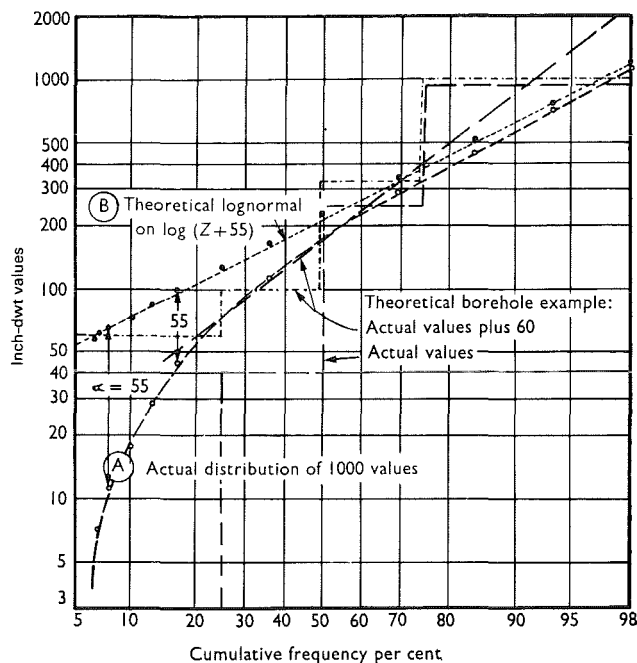

Fig. 8. Distributions of 1000 gold values from the Merriespruit mine and of four borehole values on a new property with the corresponding theoretical lognormal equivalents.

underground sampling—of less than 100%. Similarly a bias, either positive or negative, can be inherent in a set of borehole values due to, for example, contamination or core losses. No mathematical treatment of observations with an inherent but undefined bias nor an increase in the number of such observations can eliminate such a bias.

Where the population is lognormal its variance in the untransformed units (after addition of the 3rd parameter a) is given by

$$(\text{mean})^2 (e^{\sigma^2} - 1) = m^2 (e^{\sigma^2} - 1), \qquad \text{(see (3))}$$

where $\sigma^2 =$ variance of transformed units.

The error variance of the mean of a random sample of size n from any population is given by

$(1/n) \times$ population variance.

The distribution of such means for lognormal populations is *distinctly skew and not normal*: it is also not exactly lognormal, although it approaches the lognormal and eventually also the normal as n increases. As a first approximation, therefore, the error distribution can be assumed to be lognormal and its variance can be expressed in the same form as that of a lognormal distribution, namely

$$\text{error variance} = m^2 (e^{\sigma^2} - 1)/n$$
$$= m^2 (e^{\sigma_s^2} - 1).$$

Thus, σ_s^2, the logarithmic error variance (assuming lognormality), can be solved knowing σ^2 and n.

Approximate limits of error (as for the case of a normal distribution) can then be calculated. Unless the logarithmic error variance is very small ($\leqslant 0.005$), *the limits for the sample mean (as well as for other estimators) will be significantly skew and this should be borne in mind in all ore valuation procedures where lognormality applies* (see §1.11.2 and 1.11.3 below).

For stratified random perimeter sampling, which, as mentioned before, is approximately the type of sampling on South African gold mines, any sample mean can give only an estimate of the value of the perimeter concerned. The error variance for such a sample mean can be estimated as above, but, as a first approximation, the average within-strata variance should be substituted for the population variance, and the error variance, therefore, will be somewhat less than that for a random sample. The average within-strata variance will depend on the variance between the means of the strata, on the variation of the value of σ^2 from stratum to stratum, and on the correlation, if any, between the strata means and values for σ^2. From a practical point of view the effect of stratification is usually significant when dealing with the valuation of a stope face or block perimeter as such, but where these in turn are used as estimates of the value of the ore ahead of the stope face and inside the ore block respectively, the further error so introduced is usually much larger than the correction introduced by using stratified as compared with random sampling. It is in respect of this problem that the geostatistical procedures (covered in Section 2) provide significant advantages.

In the valuation of new gold mining properties from a small number of boreholes, the degree of natural stratification is normally of a low order; that is, the correlation between values is small, and the difference between stratified and random sampling, therefore, slight (Krige, 1952). This is due to the fact that on such a property the small number of deep boreholes are separated by relatively great distances. For such borehole valuations, therefore, random theory is justified and usually applied.

Owing to the shape of the log-normal distribution, with its long drawn-out tail towards the higher value categories, it is clear that even in the case where the population mean is low, there will be a definite (even if small) percentage of relatively high values. In sampling from such a population

it is inevitable that occasionally in a set of, say, 10 values, there will be one or more of these relatively high values, with the result that the arithmetic mean will be unduly influenced. A classic example of such a case was provided by the famous Free State Geduld borehole value of 23 037 inch-dwt in a set of five borehole values, of which the second highest was only 1 747 inch-dwt and the arithmetic mean 5 459 inch-dwt (1 065 inch-dwt excluding the highest value); these compare with an average value for some 84 000 metres of underground development to 1977 of 1 090 inch-dwt. This is the reason why on our gold mines the arithmetic mean of a small number of gold values, whether obtained from random sampling or systematic sampling, is known to be unreliable when the set of values includes one or more values outside the general run of values. In the past, various arbitrary methods were devised and used to 'cut' such 'anomalous' or 'freak' values. Such procedures naturally introduce an overall negative bias in the mine valuation and cannot be justified.

The knowledge of the lognormal pattern of ore value distributions allows for alternative methods of using the available data to better advantage without introducing any bias in the valuation. It can be shown mathematically that the geometric means of samples from a lognormal population are more stable — they have a much smaller error variance — than the corresponding arithmetic means. This is obvious from the following illustration. The geometric mean (say 100 cm g/t) of a lognormal sample (being equivalent to the antilog of the arithmetic mean of the normally distributed logs of the values) will be influenced no more by an exceptionally high value (say, 100 × the geometric mean value = 10 000 cm g/t) than by a low value (say, 1/100 × geometric mean = 1 cm g/t). Also, as there is a definite relationship between the geometric mean and population mean of a lognormal population, it is natural to expect that the geometric mean of a sample from such a population can be used as a basis for arriving at an estimate of the population mean which will be more reliable (that is, will have a smaller error variance) than the arithmetic mean of the sample.

The need for such an approach is illustrated by the following example of what can be expected in drilling an 'average' property in the central section of the Orange Free State field:

Average value of property — say, 4 000 cm g/t.

No. of boreholes drilled — say, 10.

On average these 10 boreholes will yield the following results (rounded off for the sake of simplicity):

Lowest value	200 cm g/t	(could be as low as a trace)
	400	,,
	800	,,
	1 200	,,
	1 600	,,
	2 000	,,
	2 800	,,
	3 600	,,
	7 400	,,
Highest value	20 000 ,,	(could be as high as 200 000 cm g/t).

It is highly significant that, on average, the highest value will be 100 times the lowest borehole value, and will represent about half of the total gold content of the 10 borehole cores combined. It requires no statistical knowledge to conclude that, if, by chance, the highest value in such a set of 10 boreholes departs from the average expected value of 20 000 cm g/t, the effect on the average of the 10 borehole values could easily swamp the effect of similar chance variations in the other 9 values. Thus, if the lowest value found is 20 cm g/t instead of 200 cm g/t, the effect will be negligible compared

Table 1

Factors for applying t'' or Sichel's t estimator to a sample of n observations

No. of observations per sample (n)		Ratio of t estimate to geometric mean of n observations											
		3	5	7	10	15	20	25	30	50	70	100	∞
	,1	1,051	1,051	1,051	1,051	1,051	1,051	1,051	1,051	1,051	1,051	1,051	1,051
	,2	1,103	1,103	1,104	1,104	1,104	1,105	1,105	1,105	1,105	1,105	1,105	1,105
	,3	1,156	1,158	1,159	1,159	1,160	1,161	1,161	1,161	1,161	1,161	1,162	1,162
Unadjusted	,4	1,210	1,214	1,216	1,217	1,218	1,219	1,220	1,220	1,220	1,221	1,221	1,221
natural	,5	1,266	1,272	1,275	1,277	1,279	1,280	1,281	1,282	1,282	1,283	1,283	1,284
logarithmic	,6	1,323	1,332	1,336	1,339	1,343	1,344	1,345	1,346	1,348	1,348	1,349	1,350
variance	,7	1,382	1,393	1,399	1,404	1,409	1,411	1,413	1,414	1,416	1,416	1,417	1,419
of	,8	1,442	1,457	1,465	1,472	1,478	1,481	1,483	1,484	1,487	1,489	1,490	1,492
observations	,9	1,503	1,523	1,533	1,542	1,550	1,554	1,557	1,558	1,562	1,564	1,565	1,568
	1,0	1,566	1,591	1,604	1,615	1,625	1,630	1,634	1,636	1,641	1,643	1,645	1,649
	1,1	1,630	1,661	1,677	1,691	1,703	1,710	1,714	1,717	1,723	1,726	1,728	1,733
	1,2	1,696	1,733	1,753	1,770	1,785	1,793	1,798	1,802	1,810	1,813	1,816	1,822
	1,3	1,764	1,807	1,831	1,851	1,870	1,880	1,886	1,891	1,900	1,905	1,908	1,916
	1,4	1,832	1,884	1,912	1,937	1,958	1,971	1,978	1,984	1,995	1,997	2,004	2,014
	1,5	1,903	1,963	1,996	2,025	2,051	2,065	2,075	2,081	2,095	2,098	2,106	2,117
	1,6	1,975	2,044	2,082	2,117	2,147	2,164	2,175	2,183	2,199	2,204	2,212	2,226
	1,7	2,049	2,128	2,172	2,212	2,247	2,267	2,280	2,289	2,308	2,315	2,323	2,340
	1,8	2,124	2,214	2,265	2,310	2,352	2,375	2,390	2,400	2,422	2,431	2,440	2,460
	1,9	2,201	2,303	2,361	2,413	2,460	2,487	2,504	2,517	2,542	2,552	2,563	2,586
	2,0	2,280	2,395	2,460	2,519	2,574	2,604	2,624	2,638	2,668	2,679	2,692	2,718
	2,1	2,360	2,489	2,563	2,630	2,691	2,726	2,749	2,765	2,800	2,813	2,827	2,858
Population	0,4	1,143	1,174	1,187	1,197	1,205	1,209	1,212	1,213	1,217	1,218	1,219	1,221
variance	0,5	1,181	1,221	1,239	1,252	1,263	1,268	1,271	1,273	1,278	1,279	1,281	1,284
known a	0,8	1,306	1,377	1,409	1,433	1,453	1,462	1,468	1,472	1,480	1,483	1,486	1,492
priori	1,3	1,542	1,682	1,746	1,795	1,834	1,854	1,866	1,874	1,891	1,898	1,903	1,916

Suggested ranges for population variance (σ^2) likely to be encountered for borehole values from goldmining properties:

On Basal Reef: 0,4 to 1,3 on the basis of (cm g/t+260) values.

On Vaal Reef: 0,5 to 0,8 on the basis of (cm g/t+174) values.

Table 2

Lower 5% limits of error for the *t* estimator for samples from a lognormal population

No. of observations per sample (*n*)		Ratio of lower 5% limit to geometric mean of *n* observations										
		3	5	7	10	15	20	25	30	50	70	100
	,1		,83	,86	,90	,93	,94	,95	,96	,98	,99	1,00
	,2		,79	,84	,89	,92	,95	,96	,97	1,00	1,02	1,03
	,3		,76	,82	,88	,93	,96	,97	,98	1,02	1,04	1,06
Unadjusted	,4		,75	,82	,89	,94	,97	,99	1,01	1,05	1,08	1,10
natural	,5		,74	,82	,89	,95	,99	1,01	1,03	1,09	1,12	1,14
logarithmic	,6		,74	,83	,90	,98	1,01	1,04	1,06	1,12	1,15	1,18
variance	,7		,74	,83	,91	,99	1,04	1,07	1,10	1,16	1,20	1,22
of	,8		,73	,83	,93	1,01	1,06	1,09	1,12	1,19	1,23	1,27
observations	,9		,74	,85	,94	1,03	1,09	1,13	1,16	1,23	1,28	1,32
	1,0		,74	,85	,96	1,05	1,11	1,15	1,19	1,28	1,33	1,37
	1,1		,74	,85	,97	1,08	1,14	1,19	1,22	1,32	1,37	1,42
	1,2		,75	,87	,99	1,10	1,17	1,22	1,26	1,36	1,42	1,48
	1,3		,76	,88	1,01	1,13	1,20	1,26	1,30	1,41	1,47	1,53
	1,4		,77	,90	1,03	1,15	1,23	1,29	1,34	1,46	1,52	1,59
	1,5		,77	,91	1,05	1,18	1,27	1,33	1,38	1,51	1,58	1,66
	1,6		,78	,92	1,07	1,21	1,30	1,37	1,42	1,56	1,64	1,72
	1,7		,80	,94	1,09	1,24	1,34	1,41	1,47	1,62	1,71	1,79
	1,8		,81	,96	1,12	1,27	1,38	1,45	1,51	1,68	1,77	1,86
	1,9		,82	,98	1,14	1,31	1,41	1,49	1,55	1,73	1,83	1,93
	2,0		,84	1,01	1,17	1,34	1,45	1,54				
	2,1		,85	1,04	1,20	1,34	1,50	1,58				
Population	0,4	,67	,76	,82	,88	,93	,97	,99	1,01	1,05	1,08	1,10
variance	0,5	,66	,76	,82	,89	,95	,99	1,02	1,04	1,09	1,12	1,14
known *a*	0,8	,64	,77	,85	,93	1,02	1,07	1,11	1,13	1,21	1,25	1,29
priori	1,3	,65	,83	,94	1,06	1,18	1,26	1,32	1,36	1,46	1,53	1,59

Table 3

Upper 5% limits of error for the *t* estimator for samples from a lognormal population

No. of observations per sample (*n*)		Ratio of upper 5% limit to geometric mean of *n* observations										
		3	5	7	10	15	20	25	30	50	70	100
Population	0,4	2,23	1,92	1,80	1,70	1,60	1,54	1,50	1,47	1,42	1,38	1,36
variance	0,5	2,53	2,15	1,98	1,80	1,74	1,67	1,62	1,59	1,51	1,47	1,44
known	0,8	3,50	2,89	2,59	2,35	2,18	2,08	2,00	1,96	1,84	1,78	1,73
a priori	1,3	5,64	4,44	3,93	3,46	3,06	2,89	2,79	2,69	2,50	2,40	2,31
	,1		1,74	1,50	1,36	1,27	1,23	1,21	1,17	1,15	1,13	1,11
	,2		2,30	1,90	1,61	1,45	1,38	1,34	1,31	1,25	1,22	1,20
	,3		2,94	2,22	1,86	1,64	1,54	1,48	1,44	1,36	1,32	1,29
Unadjusted	,4		3,67		2,13	1,84	1,70	1,63	1,57	1,47	1,43	1,39
natural	,5		4,54		2,43	2,05	1,88	1,78	1,72	1,59	1,53	1,48
logarithmic	,6		5,57		2,77	2,28	2,07	1,96	1,88	1,72	1,65	1,59
variance	,7		6,80		3,13	2,53	2,27	2,12	2,03	1,85	1,76	1,69
of	,8		8,3		3,55	2,81	2,50	2,31	2,20	1,99	1,90	1,81
observations	,9		10,0		4,00	3,11	2,74	2,54	2,40	2,14	2,03	1,93
	1,0		12,1		4,52	3,44	3,00	2,77	2,61	2,30	2,17	2,06
	1,1		14,6		5,09	3,80	3,28	3,00	2,84	2,48	2,34	2,20
	1,2		17,6		5,7	4,20	3,59	3,24	3,05	2,66	2,50	2,35
	1,3		21,2		6,4	4,6	3,93	3,50	3,17	2,86	2,67	2,51
	1,4		25,5		7,3	5,1	4,3	3,8	3,6	3,07	2,86	2,67
	1,5		30,6		8,2	5,6	4,7	4,1	3,9	3,3	3,06	2,85

with that of the highest value being 200 000 instead of 20 000 cm g/t, such deviations being almost equally likely.

If, instead, the geometric mean of the 10 values is used for comparative purposes, it is equivalent to using the antilog of the average of the logarithms of the 10 values, and changes in the two outside values from, say, 200 to 20 and 20 000 to 200 000 will leave the geometric mean unaffected.

1.11 Lognormal estimators for the mean

Statistical estimates based on the geometric mean, all derived from random sampling theory (Sichel, 1952; Krige, 1951a), are described briefly below.

1.11.1 For population variance parameter σ^2 known *a priori*, the estimator is given by

$$t'' = \text{geometric mean of } (z+a)$$

$$\times \exp\left[\frac{(n-1)\sigma^2}{2n}\right] - a. \tag{8}$$

Also, $(t'' + a)$ is lognormally distributed with error variance $\simeq (t'' + a)^2 [\exp(\sigma^2/n) - 1]$; (9) and, therefore, $\log_e(t'' + a)$ is normally distributed with variance σ^2/n. Limits of error can, therefore, be determined on normal theory for $\log_e(t'' + a)$ and hence for t'' by taking antilogs and deducting a. In Tables 1–3 factors are listed for four critical values of σ^2, for arriving at t'' and its 90 % limits of error knowing the sample geometric mean (Krige, 1951b). As indicated in Table 1, these four σ^2 values define the upper and lower limits of σ^2 for mining properties on the Vaal and Basal reefs as suggested by the available evidence.

1.11.2 For σ^2 unknown, the estimator t is given by

$$\begin{bmatrix}\text{geometric mean} \\ \text{of } (z + a)\end{bmatrix} \times \begin{bmatrix}\text{factor dependent on sample} \\ \text{variance } V \text{ and size } n\end{bmatrix} - a \tag{10}$$

where V = unadjusted variance of $\log_e (z+a)$, and
n = number of values in sample.
For n large, t becomes t' and

$$\rightarrow \exp\left[\overline{\log(z + a)} + \frac{V}{2}\right] - a.$$

The factor in (10) was tabulated by Sichel (1952, 1966) and is summarised in Table 1. This estimator is not exactly lognormally distributed but approaches lognormality and eventually normality as n increases. Based on the estimated error variance for t as defined by Sichel, and on an appropriate model for the error distribution, factors for estimating the skew 90 % limits of error have also been calculated and are listed in Tables 2 and 3 (Sichel, 1966; Wainstein, 1975). It is interesting to note that the lower limit (Table 2), which is of particular interest in ore valuation, is not very sensitive to the observed variance V for small values of n. Note that in using these three tables the constant a is added before calculating

the geometric mean and logarithmic variance and deducted again from the estimates and limits arrived at; also that the factors in Tables 2 and 3 relate to the *geometric mean of the sample and not to the* t *estimator* as in Sichel's and Wainstein's tables.

Table 4 demonstrates how skew the limits of error are — relative to the t estimator — even where the error variance is as low as 0,005. The table also illustrates the difference between the correct limits and those based on the assumption of a lognormal error distribution for the t estimator. The t estimator has a high efficiency relative to the straight arithmetic mean when the sample variance V is large (Sichel, 1952), which will always be the case where an exceptionally high value occurs in a small set of values. When V is small the advantage gained in using the t estimator rather than the arithmetic mean, is also small and frequently negligible.

1.11.3 *Examples*

1.11.3.1 A sample of 10 values drawn at random from a lognormal population with known σ^2 of 0,8, gave a geometric mean of 200 cm g/t. Then the statistical t'' estimate is 200 × 1,433 = 287 cm g/t (from Table 1, bottom section) with 90 % confidence limits of 200 × 0,93 = 186 (Table 2, bottom section) and 200 × 2,35 = 470 cm g/t (Table 3, upper section). Note the skew limits of error.

1.11.3.2 Ten borehole values on a property have a geometric mean of 200 cm g/t and a calculated unadjusted variance V, based on logs to the base e, of 0,8. From Table 1, the statistical t estimate of the population mean is 200 × 1,472 = 294 cm g/t, with 90 % confidence limits of 200 × 0,93 = 186 (Table 2) and 200 × 3,55 = 710 cm g/t (Table 3).

1.11.3.3 Four available borehole values (in cm g/t) are 3, 40, 257, and 940.

Borehole value (z)	Plus constant a of, say, 60 (z+60)	$\log_{10}(z+a)$	$\log^2(z+a)$
3	63	1,8	3,24
40	100	2,0	4,00
257	317	2,5	6,25
940	1000	3,0	9,00
Mean=310	Total =9,3	Mean =2,325	Total =22,49
			Mean =5,6225
			−5,4056 (i.e. 2,325²)
V			=0,2169 (to base 10)
			=1,1500 (to base e)
$(S^2$			=4/3×1,15=1,533)

Geometric mean   =antilog 2,325=211
From Table 1; t =211×1,68   =354
        Estimate of mean   =354 − 60 = 294 cm g/t.

Table 4

Skew limits of error for the t estimator as compared with limits based on the assumption of lognormality of the error distribution

Logarithmic error variance	0,005	0,01	0,05	0,1	0,2	0,3	0,4	0,5
Upper and lower 5% limits of error (limit/estimate of mean):								
t estimator: U.L.	1,19 (+19%)	1,27	1,72	2,19	3,13	4,13	5,26	6,71
(n=10) L.L.	0,89 (−11%)	0,86	0,74	0,66	0,57	0,52	0,48	0,45
lognormal: U.L.	1,12 (+12%)	1,19	1,48	1,78	2,32	2,87	3,47	4,13
L.L.	0,89 (−11%)	0,85	0,71	0,63	0,53	0,47	0,43	0,40

(Tables 2 and 3 have not been extended to cover the confidence limits for $n = 4$.) The graphical equivalent of this statistical estimate is shown by the dotted step-diagram and the fitted straight line in Fig. 8.

1.12 Practical considerations for borehole valuations

Apart from the statistical methods for arriving at better estimates of the average cm g/t value of a mine indicated by borehole values, the following aspects have to be considered.

1.12.1 The *lognormal model* should not be used where evidence from similar properties or from the boreholes themselves indicate that the distribution cannot be accepted as lognormal. The *t* estimators could then introduce significant biases (Koch and Link, 1971). In extreme cases the three-parameter lognormal model can also give anomalous negative lower limits, particularly where the additive constant *a* is large relative to *t*, the sample size is small, and the variance large. However, recent unpublished research by Sichel has shown that provided an additive constant is used which ensures reasonable symmetry of the transformed data, *the t estimator will have negligible or only very small biases even where the underlying distribution is distinctly non-lognormal or even J-shaped.*

1.12.2 Before attempting to estimate the mean cm g/t value, it is advisable that the borehole results on a property be studied first in conjunction with evidence from adjoining properties to detect any *regional value trends*. If such trends are clearly evident and indicate that the property can be divided naturally into two or more sections significantly different in value, such a division should be effected first and the different sections dealt with separately as suggested above. However, such a division should not be effected by a line(s) following tortuous routes between the various boreholes as this can lead to a serious overestimation of the likely grade. Where the available data allow of a geostatistical analysis (Section 2), this aspect will be covered automatically.

1.12.3 A decision is also required on whether any values should be rejected as being suspect in relation to the remaining values, for example in cases where there is a reported loss of a portion of the core on the reef-waste contact for a friable reef which is known to contain most of its gold on this contact. A general rule to be followed, irrespective of subsequent statistical considerations and of the order of the grade

disclosed, is that all values corresponding to *incomplete core recoveries* should be discarded where the responsible geologist reports that the apparent loss is such as to justify such action. Geostatistical analyses of the *deflection values* can also assist when deciding which values can be regarded as outliers.

Having determined the values to be used in the estimate, the normal procedure is to combine the individual deflection values for each borehole so as to deal with only one representative value per borehole. For this purpose the straight arithmetic mean of the values can generally not be improved upon significantly by the use of statistical procedures, since the variances within deflection areas are relatively small, the number of values dealt with is also small (usually two or three), and departures from lognormality can and do occur within such small areas.

1.12.4 Where there is any evidence of an *overall bias*, positive or negative, in the borehole values, this should be corrected.

1.12.5 If *selective mining* of the ore body to a pay limit or cut-off grade is practical, the distribution of the values of the ore blocks (or units) on which selection will be based (see §1.20), must be estimated. Again, geostatistical procedures (see §2.5.1) will cater for this problem.

1.12.6 *Dilution, losses in mining, recovery and other factors* necessary in estimating recovery grades from underground grades at the working faces must also be allowed for.

1.13 Selective mining considerations

Ross (1950) and Krige (1951a) studied in some detail the effect on the lognormal distribution of gold grades when increasing the size of the ore-unit concerned — for example, by comparing the frequency distributions for borehole (or underground channel) sample values with those of ore block grades. For the same ore body or part of an ore body, the means of these distributions must be the same, but the logarithmic variance of the former will be greater than that of the latter.

A typical example of such distributions is shown in Fig. 9. It is obvious from this diagram that for a property with an overall mean grade of 13,7 g/t and with borehole and ore block distributions as shown, the boreholes, if used directly to estimate the ore to be mined above the pay limit of 8,6 g/t, will *underestimate the tonnage* by 50 % and *overestimate the grade* by 25 %. The statistical and geostatistical procedures for estimating the appropriate ore block variance to be used in arriving at an unbiased estimate of the payable tonnage and grade will be discussed in §1.20 and 2.5.

1.14 Quality control of routine underground sampling

The bias errors inherent in the channel sampling of the gold ores led to the introduction by Sichel (1947) of statistical procedures in ore valuation on the Witwatersrand. Control procedures for the quality of such sampling was dealt with by Rowland and Sichel (1960). The basic control data comprise pairs of original and check samples taken underground in, or adjacent to, the same groove by the sampler himself, or preferably by the sampler and a senior experienced sampler respectively. Such duplicate samples are taken in about 10 % of all the sample grooves cut underground.

Conventional statistical control techniques for differences between duplicate measurements cannot be applied because the standard deviation of these differences varies with the level of the gold content of the ore samples. A study of duplicate values has shown that the proper control unit is the ratio of the original/check value on a logarithmic basis. Furthermore, if the pairs of values are first grouped in
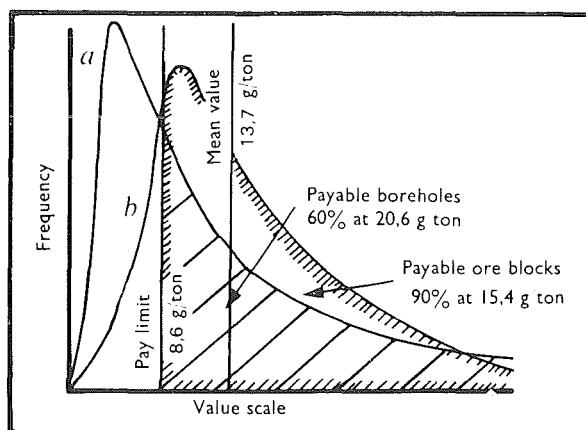
Fig. 9. Typical distributions for (*a*) borehole values and (*b*) ore blocks for a mining property.
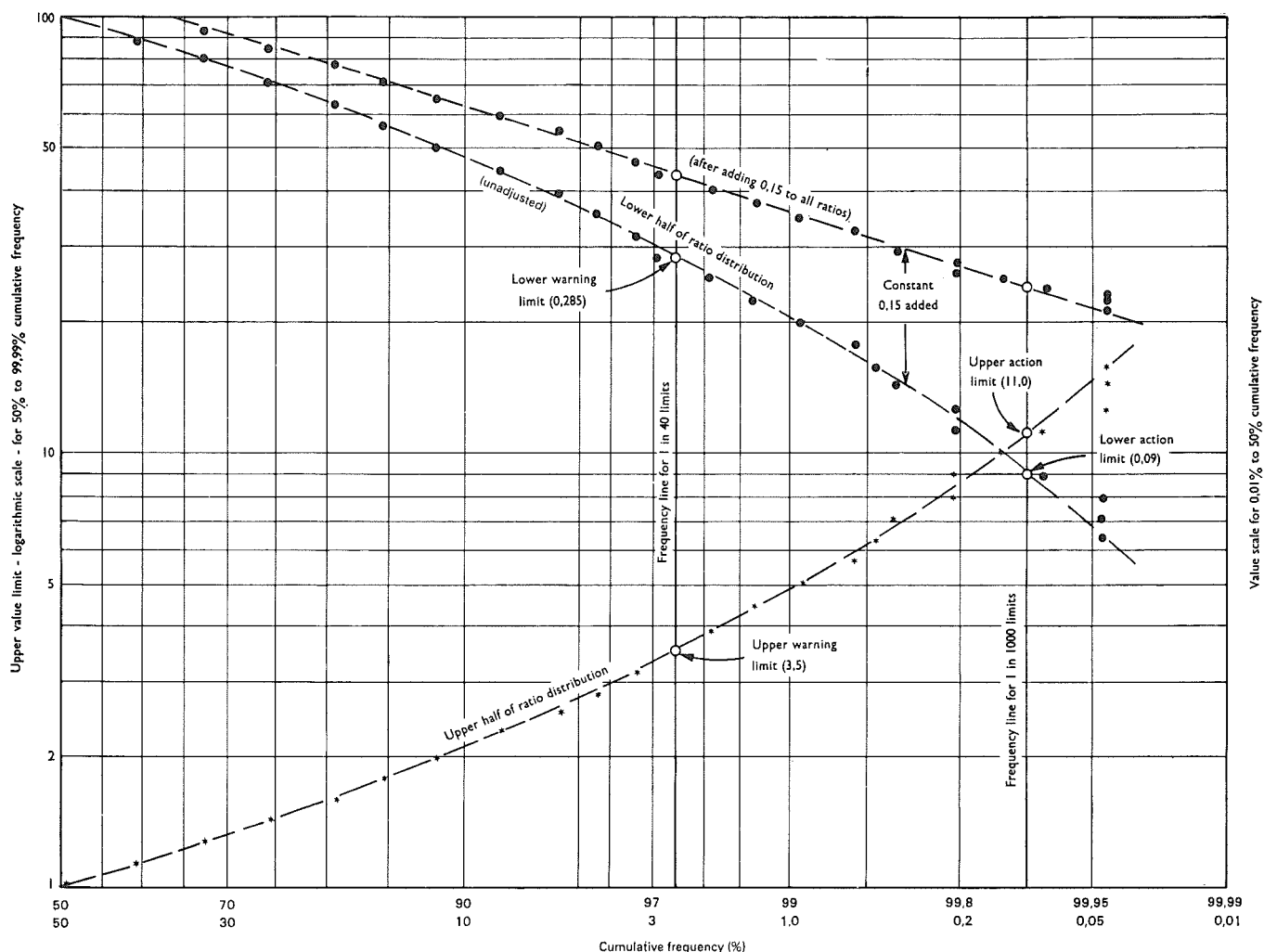
Fig. 10.　Distributions of check sampling ratios.

categories corresponding to their average gold contents, the logarithmic variances of these ratio distributions show a negligible variation from category to category. The distribution of the logarithms of these ratios is symmetrical but distinctly non-normal. The distribution is more peaked than the normal and has longer tails. Sichel developed a suitable function incorporating the auxiliary modified Bessel function of the second kind.

In practice, the author has found that where an adequate number of ratios is available, say, more than 1000, these may be plotted on logarithmic-probability paper and a smooth S curve fitted by eye to provide the required 1 in 1000 'action' and $2\frac{1}{2}$ in 100 'warning' control levels. Figure 10 shows an example of such plots. A procedure which the author has often found useful in reducing the curvature so as to facilitate the fitting of a smooth curve (sometimes reduced to a straight line), is the addition of a constant to the ratios plotted for the lower half of the distribution (Heller, 1966). The constant can be varied to reduce the observations to the closest approximation of a straight line; the necessary line or curve is then fitted, the limits read off, and the relevant constant deducted to give the estimated lower limits. The corresponding upper limits are obtained by taking reciprocals. The example of Fig. 10 has been dealt with on this basis with an additive constant of 0,15. Where the constant required approaches or exceeds 1, it becomes necessary to expand the vertical scale of the diagram. The control levels are drawn on the customary

control charts and check ratios for individual samplers are plotted on the charts for quality control purposes (Storrar, 1977). These control charts provide a relative measure of the efficiency of individual samplers and of the sampling staff as a whole both in regard to variability and bias. They also serve to disclose gross sampling errors and the likelihood of assay blunders.

1.15 Quality control of routine mine assaying

Assay control measures were developed by Coxon and Sichel in 1959. Routine assays of mine samples were then performed in the South African gold mining industry at the rate of about 13 000 per day for a total of some 50 large gold mines – equivalent to 300 assays per day per mine. Such assays are performed in bulk and an estimated correction is made for the silver contained in the button weighed after cupellation. From 5 % to 10 % of these assays are selected at random for checking purposes, a re-assay is performed and the button is parted before weighing to give a closer estimate of the fine gold content. These check assays provide, as in the case of check underground samples, a control measure for disclosing any overall bias in the routine assaying as well as an estimate of the overall relative efficiency of assaying. As for check mine samples, the appropriate control unit is again the logarithms of the ratios of original to check assays. The distribution in this case can be accepted as normal, but its variance increases

Fig. 11. Trend of logarithmic variance of ratios of original to check assays for a mine.

$$r = \frac{\text{Covariance } (x,y)}{\sigma_x \; \sigma_y} = \frac{\sigma_{xy}}{\sigma_x \; \sigma_y} \qquad (11)$$

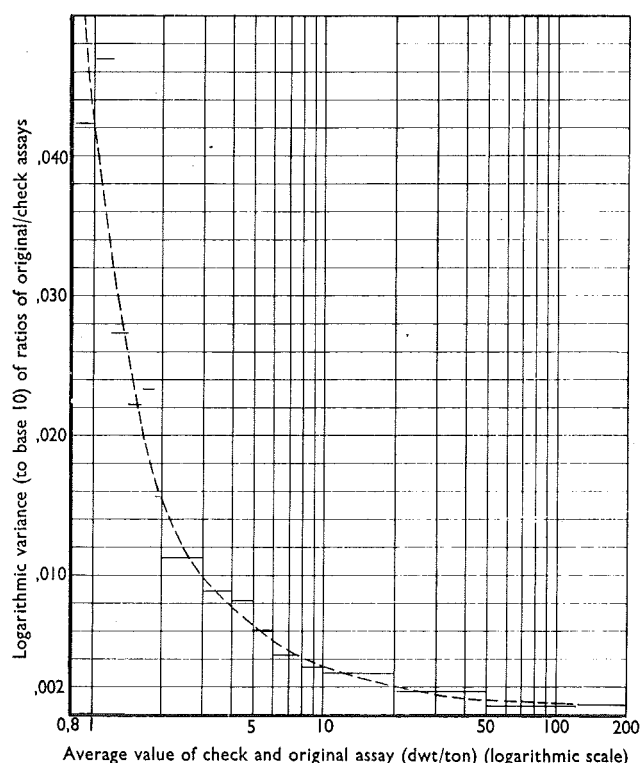$$= \frac{1}{2} \left[\frac{\sigma_x^2 + \sigma_y^2 - \sigma_{(x-y)}^2}{\sigma_x \; \sigma_y} \right] \qquad (12)$$

Equation (12) provides a convenient form for computing the correlation coefficient and also lends itself to a relatively simple graphical determination of this coefficient. All that is required is the plotting of the cumulative frequency distributions of both variables x and y and of the distribution of their differences $(x-y)$ on *probability paper* (Krige, 1962; Rendu, 1978); the variances σ^2 of these distributions are then estimated graphically and substituted in the equation. As will be seen in Section 2, the variance of the differences also forms the basis of the semivariogram used extensively in geostatistics.

In the case where $\sigma_x^2 = \sigma_y^2$,

$$r = 1 - \tfrac{1}{2} \sigma_{(x-y)}^2 / \sigma_x^2$$

also $\qquad \tfrac{1}{2} \sigma_{(x-y)}^2 = \sigma_x^2 - \sigma_{xy}.$ $\qquad (13)$

The line of regression of y on x is defined by

$$y = \bar{y} + r \frac{\sigma_y}{\sigma_x} (x - \bar{x}) \qquad (14)$$

In the case where the y and x means are identical, the regression line becomes

$$y = r \frac{\sigma_y}{\sigma_x} . x + \left(1 - r \frac{\sigma_y}{\sigma_x} \right) \bar{x}. \qquad (15)$$

Thus, *the regressed value becomes a weighted average of the variable x and the overall mean $\bar{x}$ with the weights adding up to unity.*

Figure 13 is an illustration of a correlation of two normally distributed variables x and y, their frequency distributions being shown at the bottom and left-hand sides of the graph

in the lower grade categories; that is, the variance is negatively correlated with gold grade. This is clearly illustrated by an historical example for a typical gold mine as shown in Fig. 11.

The control chart developed by Coxon and Sichel (1959) takes the form of a 'fan chart' as shown in Fig. 12. The 45° line represents the centre of the 'fan' and corresponds to perfect agreement between routine and check assays. Spaced symmetrically on either side of this line are the 99 % probability limits which serve as the required control limits. The diagram shows the original limits and the improvement achieved as reflected by the recalculated limits after a period of use of the original fan chart. These limits were determined by normal theory for the full range of grade categories using the variance-grade relationship shown in Fig. 11.

1.15.1 *Example*

For the category of 90-100 dwt/ton (1 dwt/short ton = 1,7143 g/metric ton) shown hatched in Fig. 12, the logarithmic variance (to the base 10) from Fig. 11 is 0,0008.

The logarithmic standard deviation $\quad = 0,0283.$
The 99 % limits are: 0,0283 × 2,576 $\quad = 0,0729,$
$\qquad\qquad\qquad$ antilog$_{10}$ (0,0729) $\quad = 1,183.$
$\qquad\qquad\qquad$ If original $\quad = 100$ dwt/ton,
$\qquad\qquad\qquad$ check $\quad = 100/1,183$
$\qquad\qquad\qquad\qquad\qquad = 84,5$ dwt/ton,

which corresponds to a point on the upper control line ('recalculated') on Fig. 12. In practice these fan charts (now metricated) are used on the same basis as the customary quality control charts.

1.16 Normal correlation

For two *normally* distributed variables x and y the correlation coefficient

Fig. 12. Fan chart for the control of check assaying, with 99% probability zone for duplicate pairs of assays.
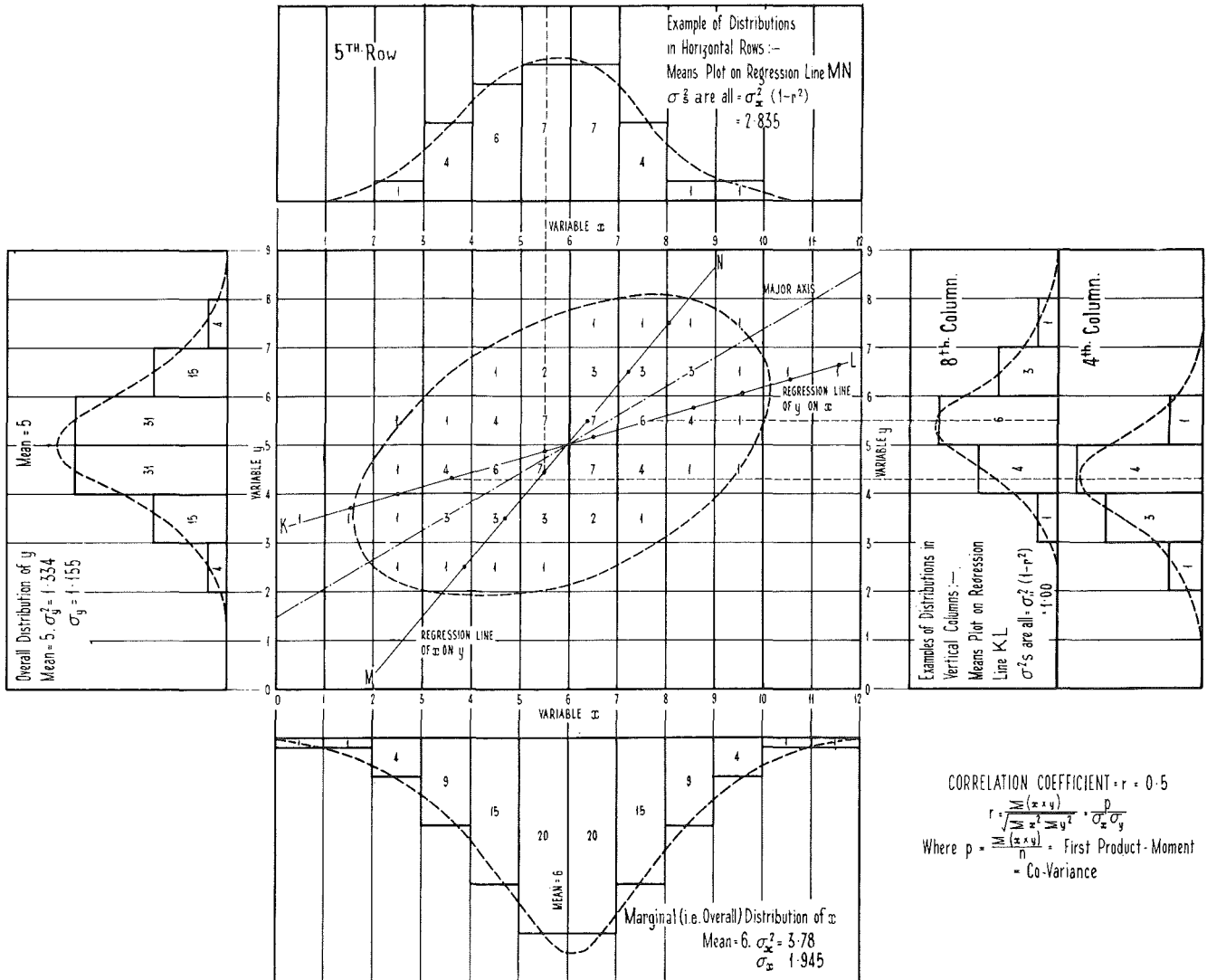
Fig. 13. Main features of a normal correlation of two variables.

respectively. The figures on the main graph represent a *correlation table* of a specimen sample of 100 pairs of values drawn from the distributions of x and y, with the relevant frequencies shown in each category block — thus, of the 100 pairs of values, six fall in the category $x = 4$ to 5, $y = 4$ to 5. The means of the various columns of y values thus formed plot along the straight line KL, this being the *line of regression* of y on x, that is, the average trend of y relative to x. Similarly, the mean of the horizontal rows of x values plot along MN, which is the regression line of x on y.

The intensity of frequencies, if contoured, will follow elliptical curves with a major axis as shown between lines KL and MN. For perfect correlation all the observations will plot along this major axis and the two regression lines will also coincide with it. If there is no correlation, the two regression lines will be parallel to the x and y-axes — that is, they will be perpendicular to each other.

In the example shown the correlation is positive — larger values of y are generally associated with higher values of x. For negatively correlated variables the reverse will apply and the major axis of the ellipse will have a negative slope.

Another feature of the normal correlation surface is that the variances in the columns (or *arrays*) of y values are all equal,

and similarly the variances for the arrays (or rows) of x values are all identical. Examples of the distributions within rows and columns are shown on the diagram.

The *conditional* variances in the arrays of x and y values are related to the overall (or *marginal*) variances of x and y as follows:

Variance in arrays of x values $= \sigma_x^2 (1 - r^2)$.

(16)

Variance in arrays of y values $= \sigma_y^2 (1 - r^2)$.

The correlation coefficient can vary from -1 (perfect negative correlation) through zero (no correlation) to $+1$ (perfect positive correlation). When $r = \pm1$ (corresponding to perfect correlation) the variances in the arrays will be zero, in which case the observations will all lie on the coinciding lines of regression indicating a strict linear relationship between x and y. Also, for zero correlation $(r = 0)$, the variances in the arrays will be equal to the corresponding overall or marginal variances for x and y respectively. With the lines of regression being parallel to the x and y-axes, the frequency intensity will then change from the elliptical to a circular pattern (provided the axes are scaled in units of the

marginal standard deviations); there will then be no trend of x against y or vice versa.

It should also be noted that, where two variables are correlated as shown in Fig. 13, *the average value of y corresponding to any specified value for x* (as indicated by the regression line of y on x — line *KL*) *in the lower value categories of x, will always be higher than what it would have been under perfect correlation*, with the reverse position holding in the higher value categories of x. A similar position will naturally hold for the regression of x on y.

1.17 Lognormal correlation

For two lognormally distributed variables the equations for normal correlation will also apply provided logarithms of the values (or of the values plus a constant) are used. The centre of the ellipse will, in this case, however, correspond to the means of the two overall (or marginal) distributions of the logarithms (that is, to the marginal geometric means of the two lognormal variables). Similarly, the straight regression lines through this centre point will represent the trends of the means of the logarithms of the values in the relevant arrays (that is, the trends of the geometric means of the various columns and rows of the lognormal variables). If, as is normally the case, an estimate is required of the regression of lognormal variable y on lognormal variable x (as distinct from the straight line regression of log y on log x), the displacements between the relevant means and geometric means within the arrays therefore have to be taken into account. This displacement will correspond to the displacement between the means and the geometric means of the conditional lognormal distributions in the relevant arrays (Krige, 1951a, 1962). The regression of y on x will therefore be defined by:

$$\log z = \overline{\log y} + r\,\frac{\sigma_y}{\sigma_x}\,(\log x - \overline{\log x}) + \tfrac{1}{2}\,\sigma_y^2\,(1 - r^2)$$

or

$$\log z = \log \bar{y} + \tfrac{1}{2}\,r\,\sigma_y\,(\sigma_x - r\,\sigma_y)$$
$$+ r\,\frac{\sigma_y}{\sigma_x}\,(\log x - \log \bar{x}), \qquad (17)$$

where z = regressed estimate of y based on the known x value,

y and x are the two lognormal variables (after addition of a constant if necessary),

$\bar{y}$ and $\bar{x}$ are their respective means,

σ_y^2 and σ_x^2 are their respective marginal logarithmic variances, to base e,

r is the correlation coefficient of $\log_e y$ and $\log_e x$, and

$\tfrac{1}{2}\sigma_y^2(1 - r^2)$ is the displacement between the log means and log geometric means in the y arrays (see (1) and §1.18).

The following example of a lognormal correlation will illustrate the principles involved. On a mine a pre-metrication study was made of the frequency distribution of the values of available 150 ft (46 m) stope panels (current and old) and of the correlation between these values for pairs of panel faces 30 ft (9 m) apart on strike. It was found that, after adding a constant 20 inch-dwt (87 cm g/t) to all the panel values, the values so adjusted could be accepted as lognormally distributed. The required correlation table was therefore prepared as shown in Fig. 14 on the basis of scales of log (inch-dwt + 20) for the available 557 pairs of panel values. The

correlation table is a straightforward two-way classification of the pairs of values. In classifying the first of the two panel values from each pair into the value categories shown along the x-scale, 32 were found to lie between 92 and 158 inch-dwt, and on classifying along the y-scale the corresponding second panel value from these 32 pairs (the values of the panel faces lying 30 ft ahead), it was found that 1 panel had a value between 51 and 92 inch-dwt, 8 had values between 92 and 158 inch-dwt, 17 had values between 158 and 262 inch-dwt, and the remaining 6 had values between 262 and 427 inch-dwt. The averages of the values in this column of values were found to be:

for the x values 133 inch-dwt, and for the y values 206 inch-dwt.

This suggests that a panel face of 133 inch-dwt will, on average, have, at a distance of 30 ft ahead, ore which will yield a face value of 206 inch-dwt. Had there been perfect correlation between panel faces 30 ft apart, common sense dictates that these two averages would have been identical, and the factual position, therefore, again illustrates this very important basic feature of correlation and regression analysis as applied to mine valuation, that *any imperfect measurement or prediction of the value of ore along a stope face or within an ore block will on average understate the true value of such ore in the lower grade categories, and overstate it in the higher grade categories.* In the above example of 32 values the first panel value in a pair can be regarded as the imperfect estimate of the panel value of the ore lying 30 ft ahead. The overstatement in the higher grade categories will be evident from, for example, the second last column in Fig. 14, where 25 panel values averaging 788 inch-dwt have ahead of them (at a distance of 30 ft) 25 faces averaging only 572 inch-dwt.

The corresponding averages for the other columns were similarly determined and are also shown on the diagram. Plotted graphically these points clearly follow a straight line trend, the departures from the straight line shown being well within the chance variations due to the limited number of observations. This trend of average y values in the columns corresponding to specified x values is the regression of y on x determined directly from the arithmetic means of the

							Column averages (inch-dwt)
75	133	213	336	519	788	1170	for x values
153	206	248	350	452	572	490	for y values
5	32	153	195	145	25	2	No (Total = 557)

Frequency grid (y rows from 1758 top down to 51; x columns: 75, 133, 213, 336, 519, 788, 1170):

y \ x	75	133	213	336	519	788	1170
1758						1	
1102				4	14	6	
688			9	47	62	10	1
427		6	47	94	52	6	1
262	1	17	75	46	15	1	
158	3	8	17	4	2	1	
92	1	1	5				
51							

y (scale of log (inch-dwt + 20)) — Sampled value of panel face 30 ft ahead (inch-dwt)

Sampled value of panel face (inch-dwt)
x (scale of log (inch-dwt + 20))

Trend of average y values for specified x values, i.e. regression line of y on x based on calculated correlation coefficient (observed averages shown thus •)
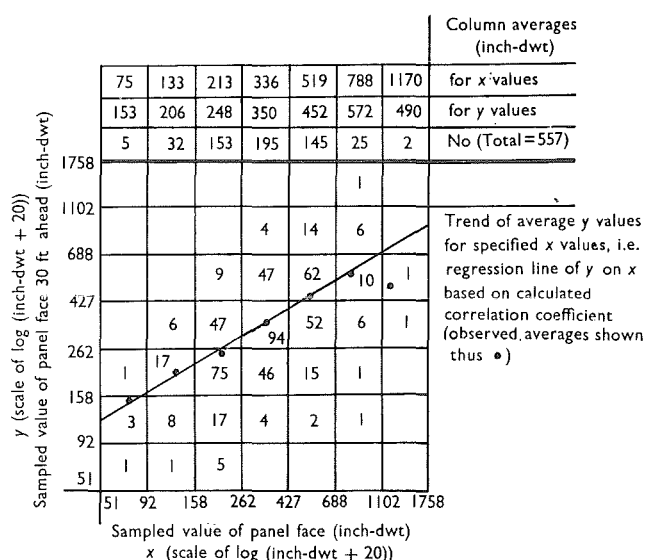
Fig. 14.   Correlation between panel face values 30 ft apart and hence the value on average of the ore 30 ft ahead of a panel face of any specified value.

Fig. 15.   Illustration of the effects of regression on gold ore block values.

values in the columns. Statistical correlation and regression techniques provide the mathematically calculated solution for this trend, and in effect the application of the above regression equation (17), therefore, will yield the theoretically correct equation for the straight line which fits the column averages best.

On a calculated basis for this example:

$$\sigma_x^2 = \sigma_y^2 = 0,21 \; ; \bar{x} = \bar{y} = 378,$$

$$r = 0,55.$$

The regression formula is

$$\log_{10} (y + 20) = 1,171 + 0,55 \log_{10} (x + 20),$$

and this agrees closely with the plotted line.

1.18 Ore reserve regression curves

The phenomenon of the undervaluation of low-grade ore and the overvaluation of high-grade ore as referred to above, was known from the early years of gold mining in South Africa (Ross, 1950) but was not demonstrated to be an unavoidable regression effect until 1951 (Krige, 1951a). It was observed in the early tonnage and grade returns on all gold mines when the monthly results based on current stope sampling were compared with the corresponding orthodox ore reserve block values in the various grade categories.

The *block factors* in the various ore reserve grade categories — the ratio of gold called for from current stope sampling to that called for from the corresponding ore reserve block values — in general exceed 100 % in the lower grade blocks and fall below 100 % in the higher grade block categories. Bearing in mind that ore block values are imperfect estimates

of the value of the ore inside such blocks — and that the subsequent current stope samplings inside these blocks (if accumulated over a sufficiently long period of time and over large numbers of blocks per grade category) provide close estimates of the *actual* average sampling value of the ore inside these blocks — it should be clear that this trend of block factors corresponds to the regression of the internal on the peripheral block value with the current stope value and the ore reserve block value as the respective correlated lognormal variables.

Figure 15 provides a simplified graphical explanation of the phenomenon. It is assumed that the actual correct gold block values are distributed lognormally with a logarithmic variance σ^2 of 0,25 and mean of 10 g/t along the y-axis, and that in valuing these blocks on limited peripheral (or internal) samplings the errors incurred are also distributed lognormally with logarithmic variance 0,15. This results in a lognormal distribution of observed block values along the x-axis with log variance $= 0,25 + 0,15 = 0,40$ (see (34)). As the scales along the two axes are logarithmic, the bivariate distribution of actual and observed values is normal. The regression of observed on actual values (of x on y) is taken as linear along the 45° line AB, as the errors committed in the valuation of individual blocks are assumed to be unbiased. But from standard regression theory the regression of actual on observed values (y on x) will have a flatter slope along line CD. Thus the actual average of blocks with observed values of, say, $2\frac{1}{2}$ g/t (that is, $x = 2\frac{1}{2}$) will be 4,3 g/t (y value on line CD corresponding to $x = 2\frac{1}{2}$), thus demonstrating again that where valuation errors are committed (unbiased with respect to an individual ore block), the combination of blocks valued in a low-grade category will be undervalued. The reverse will, of course, be evident for blocks valued in the high-grade categories.
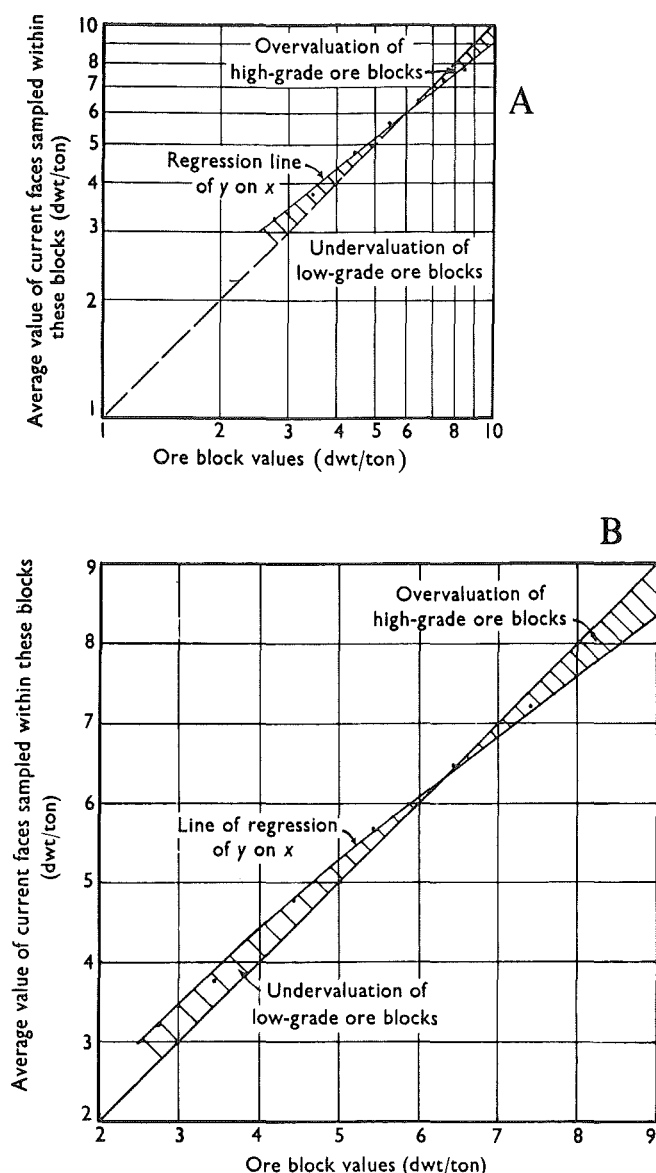
Fig. 16. Example of regression between ore reserve block values and values indicated by subsequent sampling of stope faces within such ore blocks.

The extent of the biases so introduced will depend on the variability of the ore values and on the error variance of block valuations, and the latter in turn will be determined by the quality and intensity of underground sampling and the level of correlation between the peripheral values and values inside the ore blocks. Biases would typically be up to some 20 % but could be as high as 50 % (Storrar, 1966).

A further typical, practical example of this phenomenon from another gold mine is illustrated in Figs 16a and 16b. Figure 16a shows on logarithmic scales the regression of the follow-up stope values on the imperfect ore block values in the summarised form of the column means only — of the regression of stope on ore block values. Figure 16b shows the same position on arithmetic scales, and the under and the overvaluation of the ore blocks in the lower and higher grade categories respectively are clearly evident.

Where a mine has maintained a reasonably steady grade position over a number of years, this average regression

trend can readily be established from the gold contents based on stope and block valuations respectively and used in calculating the block factors in the individual grade categories. These contents, expressed in the form of inch-dwt or cm g/t values, are plotted (as in Figs 16a and 16b) on double logarithmic paper (if no additive constant is required) or on specially prepared graph paper based on the required scales of log(block value + constant) along the x-axis and log(stope value + constant) along the y-axis. The straight line of best fit for these plotted points will then represent the required *ore reserve regression curve*, which can be used for adjusting the values for the new ore reserve blocks. Mathematically, the best fitting straight line can be obtained by applying the method of 'least squares' with the individual points (corresponding to the averages in the categories of estimated block grades) weighted proportionally to the corresponding fathomages or square metres stoped out.

As shown in §1.20, the parameters for the regression line are not affected by the fact that the follow-up y values are themselves usually subject to error, i.e. that *actual* follow-up values are not known.

Such regression curves are now in general use on the South African gold mines, except on mines where the more advanced kriging procedures are used (see Section 2). It should be noted that where changes in average grade, sampling intensity, block sizes, and ore variability occur from year to year, the straight application of historical regression curves could introduce significant biases (Krige, 1962, Section IV, §3); these would be avoided with kriging procedures. It should again be stressed — see (15) — that such regressed estimates of the values of ore blocks can be regarded as weighted averages of the block estimates and the overall mean value of all the ore blocks. If the mean of all the ore blocks is seen as the value of a complete peripheral zone or aureole around the ore block, the regression of block values becomes the *first elementary form of the kriging procedure*, as introduced by the author and subsequently developed in its full context by Matheron (1960, 1971).

1.19 Effects of regression on total ore reserves

These effects were partly demonstrated in Fig. 15. A further historical example is provided by the case of the B reef on the Loraine mine (Krige, 1962). The unit for stope valuations was accepted as a 120 ft (37 m) panel face and the ore selection unit as an area of 120 ft (37 m) by 80 ft (24 m) in depth ahead of a panel face. A correlation table was prepared for 838 panel values and the means of 838 corresponding sets of 3 panel values, each in respect of faces lying within the 80 ft area ahead of each original panel face, the latter being accepted for purposes of this analysis as corresponding closely to the values of the 80 ft areas concerned. The logarithmic correlation coefficient was determined as well as the corresponding line of regression of the value (y) of the intact ore ahead (average of 3 panel values ahead) on the original panel value (x). This regression line is shown as AB in Fig. 17.

A horizontal line through the official pay limit of 4,1 dwt/t (7 g/t) on the y-scale of actual intact ore values clearly divides the area inside the ellipse into an actually payable upper half (representing 46,8 % of the total ore) and an actually unpayable lower half. With perfect valuation this perfect classification would be possible, but in practice only the imperfect panel values along the x-scale are available.

On the basis of these panel values it is clear that *the stoping of some unpayable ore is unavoidable* — even faces with grade estimates as high as 10 dwt/ton will have ahead of them blocks of intact ore ranging in actual value from about 2 dwt/ton (unpayable) to 13 dwt/ton (average 6,2 dwt/ton).
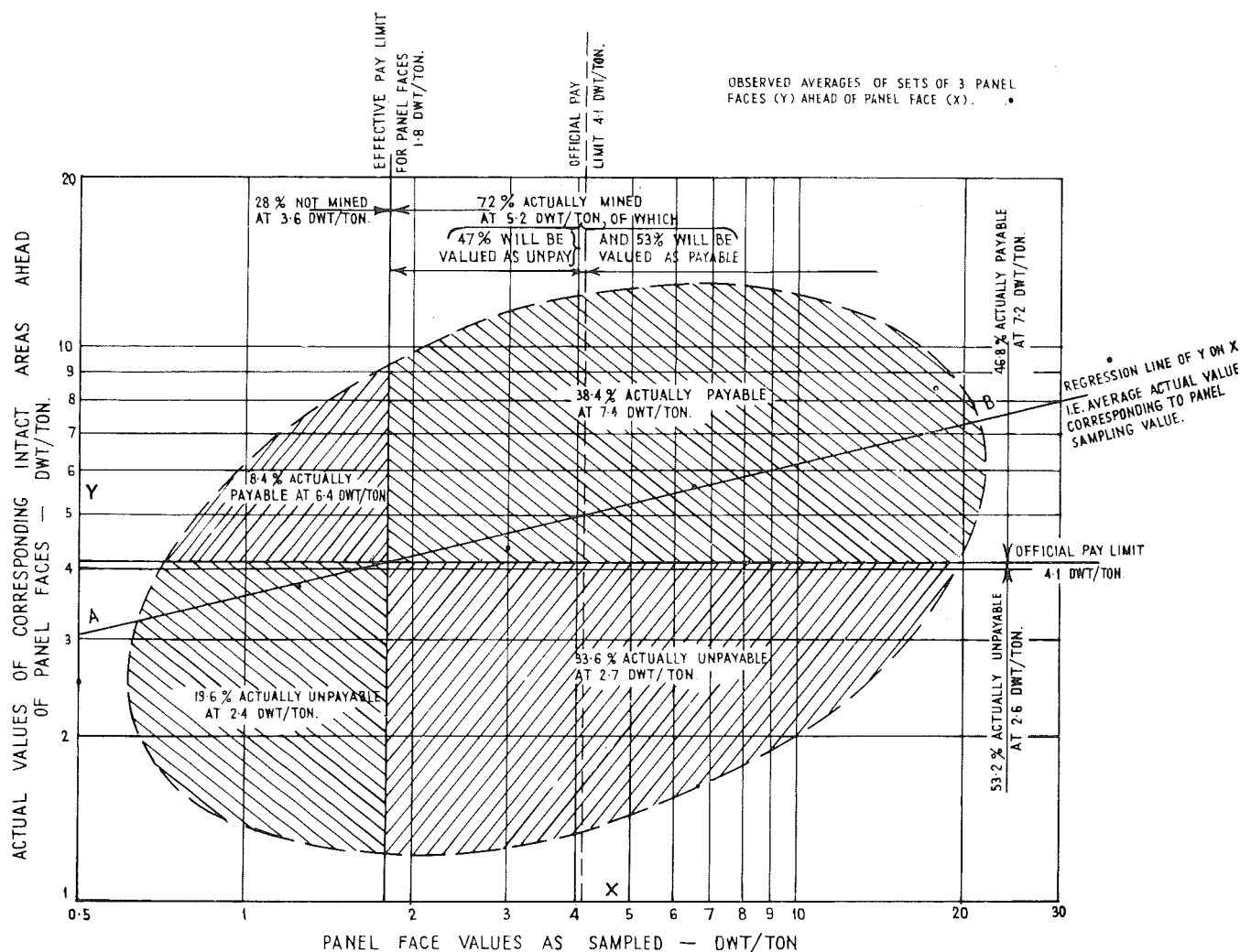
Fig. 17. Correlation surface illustrating the relative efficiencies of an effective pay limit and the official pay limit in classifying unregressed block values.

Application of the official pay limit to the available panel values (x) will show a total payability of only some 38 % (to the right of the vertical line through 4,1 dwt/ton). However, some of the ore below this pay limit can still be mined as payable on average owing to the regression effect. The optimum selective position is indicated by panel face values down to 1,8 dwt/ton, which will have ahead of them blocks of intact ore averaging 4,1 dwt/ton in actual value, which is the official pay limit. Panel faces less than 1,8 dwt/ton will in some cases still have payable ore blocks ahead of them, but the unpayable ore blocks will outnumber and outweigh them to give an unpayable overall average. The required *effective pay limit* is therefore 1,8 dwt/ton.

The ore selected for mining above this effective pay limit, therefore, is represented by the right-hand side of the ellipse (72 % of the total ore of which 33,6/72 = 47 % will in fact be unpayable and 38,4/72 = 53 % will be payable). Ore left unmined is represented by the left-hand side of the ellipse (28 % of the total ore, of which 30 % will be payable and 70 % unpayable). Application of regression will in this case therefore *increase the ore reserves from 38 % to 72 %*.

Application of this effective pay limit will maximize the profits to be earned on selective mining based on panel values in spite of the relative inefficiency of the operation. If the block values are all first regressed, the same result will be

achieved by using the official pay limit instead of the effective pay limit. The position can be further improved only by improving the standard of stope sampling, by reducing the size of the ore selection unit — the block size — through, for example, predevelopment on a smaller grid and/or by even more efficient valuation procedures (see §3.3.6).

This example on B reef is perhaps an extreme case because of the very low correlation between panel values and values of intact ore blocks ahead of the panels. This is due to the highly erratic nature of the values in this ore body, or, in geostatistical terminology, because the *nugget effect* forms such a large part of the total variance. The effects have also been exaggerated because the averages of the sets of 3 panel faces within the 120 ft × 80 ft blocks are still only imperfect estimates of the actual block values, although this aspect does not affect the definition of the regression line itself (see §1.20). *This regression effect will be present in all imperfect ore reserve valuations for all minerals* but could be insignificant in some cases. Where it is significant, regression adjustments and preferably kriging procedures should, therefore, be applied.

1.20 Effective block variance

For the model shown in Fig. 15 the following relationships can be derived:

Let x = unregressed value = estimated block value,

y = actual block value,

z = regressed block value,

$\bar{x} = \bar{y} = \bar{z}$ = mean value of ore body.

σ_{xe}^2 = logarithmic error variance of x values,

σ_{ze}^2 = logarithmic error variance of z values,

σ_y^2 = variance of natural logs of y values,

$\sigma_x^2 = \sigma_y^2 + \sigma_{xe}^2$ = variance of natural logs of x values.

Then: r = correlation coefficient between the normalised x and y values.

$$= \sqrt{\left(1 - \frac{\sigma_{xe}^2}{\sigma_x^2}\right)} = \sqrt{\frac{\sigma_x^2 - \sigma_{xe}^2}{\sigma_x^2}} = \frac{\sigma_y}{\sigma_x}.$$

$$\text{Cov}_{xy} = \sigma_{xy} = r\sigma_x\sigma_y = \sigma_y^2 \qquad (18)$$

Regression:

slopes of lines: y on x: $r\dfrac{\sigma_y}{\sigma_x} = \dfrac{\sigma_y^2}{\sigma_x^2}$ (18a)

x on y: $r\dfrac{\sigma_x}{\sigma_y} = 1$ (18b)

Also $\log z = \log \bar{y} + \frac{1}{2} r\sigma_y(\sigma_x - r\sigma_y)$

$$+ r\frac{\sigma_y}{\sigma_x}(\log x - \log \bar{x}). \qquad (18c)$$

Error variance of z (i.e. conditional variance in y arrays):

$$\sigma_{ze}^2 = \sigma_y^2(1 - r^2) = \sigma_y^2 - \sigma_z^2 = \frac{\sigma_y^2 \sigma_{xe}^2}{\sigma_x^2} \qquad (19)$$

and variance of z (i.e. variance of means of the y arrays along regression line CD):

$$\sigma_z^2 = r^2\sigma_y^2 = \frac{(\sigma_x^2 - \sigma_{xe}^2)^2}{\sigma_x^2} = \frac{(\sigma_y^2)^2}{\sigma_x^2}$$

$$= \sigma_y^2 - \sigma_{ze}^2. \qquad (20)$$

These formulae assume that the third additive parameter for the lognormal distribution can be disregarded or that the x and y values already have the constant added.

It should be noted that *the logarithmic variance of the regressed block values* σ_z^2 *is lower than that of the actual block values* σ_y^2 and allowance should be made for this when estimating the likely percentage and grade of ore blocks to be mined above a specified pay limit for an undeveloped new mine or section. This feature will also be evident when using the more sophisticated kriging procedures and will be referred to again in §2.5. In these procedures (18) will also be of particular significance: *the covariance between an unregressed estimate of a block value based on sampling of the block itself and the actual block value is equal to the variance of the actual block values.*

Note that these conclusions based on (18)–(20) are strictly valid only for the case where the regression of x on y — of the unregressed estimates on the actual block values — has a slope of unity [see (18b)], and the variance of the estimates thus comprises the variance of the actual block values plus the error variance of the estimates. Equations (19) and (20) will therefore apply strictly only where the physical sampling is practicable over the whole interior of the block and not only along its periphery or over a selected part, so that, as the

number of ore samples approaches infinity, the valuation will approach the actual block value. This condition will be satisfied in sampling the whole ore block on a random, stratified random or systematic grid (with a random start) basis. In the case where this condition is not satisfied and sampling is confined, for example, to the block periphery, the following will apply:

Let σ_p^2 = variance of actual peripheral values p,

r_1 = correlation coefficient for actual internal block values y and these p values,

σ_e^2 = error variance of x values as estimates of the corresponding p values,

x = unregressed block values based on the observed values of the peripheries with actual values p,

and y, z, r, σ_{ze}^2, σ_y^2 and σ_x^2 have the same meanings as before.

Then, in the correlation of y and p values, conditional variance in arrays of p values $= (1 - r_1^2)\,\sigma_p^2$. On superimposing the peripheral error variance σ_e^2 to change the p values to x values, the correlation of the y and x values becomes:

Conditional variance of $x = (1 - r_1^2)\,\sigma_p^2 + \sigma_e^2$

$$= (1 - r^2)\,\sigma_x^2$$

and $\qquad \sigma_x^2 = \sigma_p^2 + \sigma_e^2.$

Then $\qquad r^2 = 1 - \dfrac{(1 - r_1^2)\,\sigma_p^2 + \sigma_e^2}{\sigma_p^2 + \sigma_e^2}$

and $\qquad r = \dfrac{r_1\,\sigma_p}{\sigma_x}.$

Also, $\sigma_{xe}^2 = \sigma_{(x-y)}^2 = \sigma_x^2 + \sigma_y^2 - 2r\,\sigma_x\,\sigma_y.$

Equations (18), (19) and (20) then change to:

$$\text{cov}_{xy} = \sigma_{xy} = r\,\sigma_x\,\sigma_y = r_1\,\sigma_p\,\sigma_y, \qquad (18d)$$

$$\sigma_{ze}^2 = \sigma_y^2(1 - r^2) = \frac{\sigma_y^2\,(\sigma_x^2 - r_1^2\,\sigma_p^2)}{\sigma_x^2}, \qquad (19a)$$

$$\sigma_z^2 = \sigma_y^2 - \sigma_{ze}^2 = \frac{r_1^2\,\sigma_y^2\,\sigma_p^2}{\sigma_x^2}. \qquad (20a)$$

Note that (18d), (19a) and (20a) will be identical to (18), (19) and (20) respectively if $r_1 = 1$ and $\sigma_p^2 = \sigma_y^2$, which will be the case under conditions of random, stratified random or systematic grid sampling of the ore blocks.

It should also be noted that the effective block variance as well as the regression formula for follow-up values (y) on block valuation (x), i.e. (17) and (18c), is not affected if the follow-up values are not *actual*, but subject to a logarithmic error variance σ_{ye}^2 ; σ_y^2 then becomes $\sigma_y^2 + \sigma_{ye}^2 = \sigma_{y1}^2$, r reduces to $r_x = \gamma / \sqrt{1 + \sigma_{ye}^2/\sigma_y^2} = \sigma_y^2/\sigma_x\sigma_y$; slope parameter r σ_y/σ_x becomes r $\sigma_y^1/\sigma_x = \sigma_y^2/\sigma_x^2$, i.e. the same as (18a), and the effective block variance σ_z^2 now $= r^2_x\,\sigma_{y1}^2 = (\sigma_y^2)^2/\sigma_x^2$ i.e. also unchanged from (20).

1.20.1 *Grade and tonnage estimates for new properties*

It is evident from the foregoing that in framing grade and tonnage estimates for an ore body, it is essential, firstly, to estimate the mean grade of the ore body with the appropriate — usually skew — limits of error as efficiently as possible

and, secondly, to use the correct model for the estimated value distribution of the ore units (or blocks) on which any selective mining is to be effected. For the lognormal case, the first step requires the use of the t estimator (§1.10–1.12) and the second step, which involves the estimation of the effective block variance (see §1.18–1.20; also §2.5 and 2.11 if kriging is to be used), requires:

(i)   the specification of the size of ore block on the basis of which selective mining decisions will eventually be taken (in the case of a massive ore body the final selection could be on the evidence of closely spaced blast holes and the selection unit will then be relatively small);

(ii) an estimate of the variance σ_y^2 (and distribution model) of the actual values of the ore units in (i) (see §2.5); and

(iii) an estimate of the error variance, for example σ_{Ze}^2, likely to apply to the valuation of these ore units during the selective mining of the ore, and taking proper account of the expected systems of sampling and valuation of the ore units (see §2.5 and 2.11) after elimination of all biases due to the regression effect.

This procedure will avoid serious bias errors in ore reserve estimates for new mining properties such as referred to by Marechal (1977). For operating mines, regression and kriging techniques can show substantial financial advantages (see §3.3.6 and Krige 1981).

1.21 Regression on regional means

Investigations into the covariance and correlation levels between pairs of individual values (and pairs of individual face values) at various distances apart for the two members of each pair (Krige, 1963, 1968), led naturally to the conclusion that the closer together values were, the higher the level of correlation between them; and hence that, in the valuation of an ore block, the values outside the periphery will be correlated at progressively lower levels with the actual internal block value as their distances from the block increase, and should, therefore, also carry progressively lower weights. This logic suggested that the efficiency of valuation would be improved by using all available data within, on the periphery of, and outside an ore block on the basis of progressively decreasing relative weights. The second step in the evolution of the kriging procedures thus involved the introduction of a more local or regional mean, based on areas of 700 000 sq ft (65 000 m²) (Krige and Ueckermann, 1963). Such a more local aureole, being based on a large number of available values, could normally be valued reasonably accurately, and with the necessary correlations established for data in mined-out sections of the mine, the relative weights to be attached to the peripheral block value, the value of the local aureole and the overall mine average (as a second aureole) could be established and applied to new block valuations. This procedure was in use for some two years on a few of the gold mines and proved to be a significant improvement on the straight regression procedure; it led naturally to the kriging procedures discussed in Sections 2 and 3.

Practical Geostatistics for the Lognormal-de Wijsian Model

2.1 General

The term geostatistics was introduced by Matheron (1960) to cover the development and application of mathematical and statistical models which take specific account of the *spatial structure* of a *'regionalised'* variable. In ore valuation the relative positions of available ore values within an ore body or portion thereof, can only be ignored in the unlikely case where there is no significant pattern of association or correlation between them — that is, if they do not disclose any significant spatial structure. A structure might not be evident in the available data, e.g. where the spacing of the available values is relatively wide such as in surface drilling on a new deep gold mine. However, practical selective mining considerations could dictate the need for eventual underground valuation on a scale much smaller than that represented by this spacing and the possibility of a significant structure on such a smaller scale cannot then be ignored.

Some spatial structure must be present wherever the average variability among the values of check ore samples from the same, adjacent or nearby grooves or of borehole cores from the deflections in the same borehole, is lower than the overall variability for ore values within the whole ore body. Another practical check on the presence of a spatial structure is where it is evident that there is or will be a significant variation in the ore grades for individual ore blocks. Any correlation between peripheral and follow-up internal ore block values also provides evidence of a spatial structure, and hence the regression techniques covered in §1.16–1.21 can be accepted

as the first application of the geostatistical procedures.

The spatial structure of a regionalised variable can be studied and measured by the pattern, usually a decreasing function, in the level of correlation or of the covariance between pairs of values as the distance or lag between the values increases, that is, by the *correlogram* or *covariogram* (Krige, 1968, 1969, 1976; and Fig. 18). In geostatistics, for various practical and theoretical reasons (Matheron, 1960, 1971), the measure most commonly used is the *semivariogram*, which increases as the lag increases. The semivariogram and covariance values are interrelated and for the lognormal-de Wijsian pattern used extensively for gold ore valuation both measures as well as their relationship can be defined on a fairly straightforward basis. Provided the underlying assumptions are realistic, the choice of measure — correlogram, covariogram or semivariogram — will be dictated mainly by convenience.

Paragraph 2.2 will cover some general geostatistical terms and §2.3–2.12 will deal mainly with the theoretical side of the lognormal-de Wijsian model for spatial structures. Wherever the relevant variable lends itself to normalisation via the 3-parameter lognormal distribution (see §1.9) the de Wijsian semivariogram model is also likely to be suitable and will provide substantial advantages in application. Various practical applications will be covered in Section 3. Although confined to the lognormal-de Wijsian model, much of the theory covered and the general approach is applicable to all spatial structures.
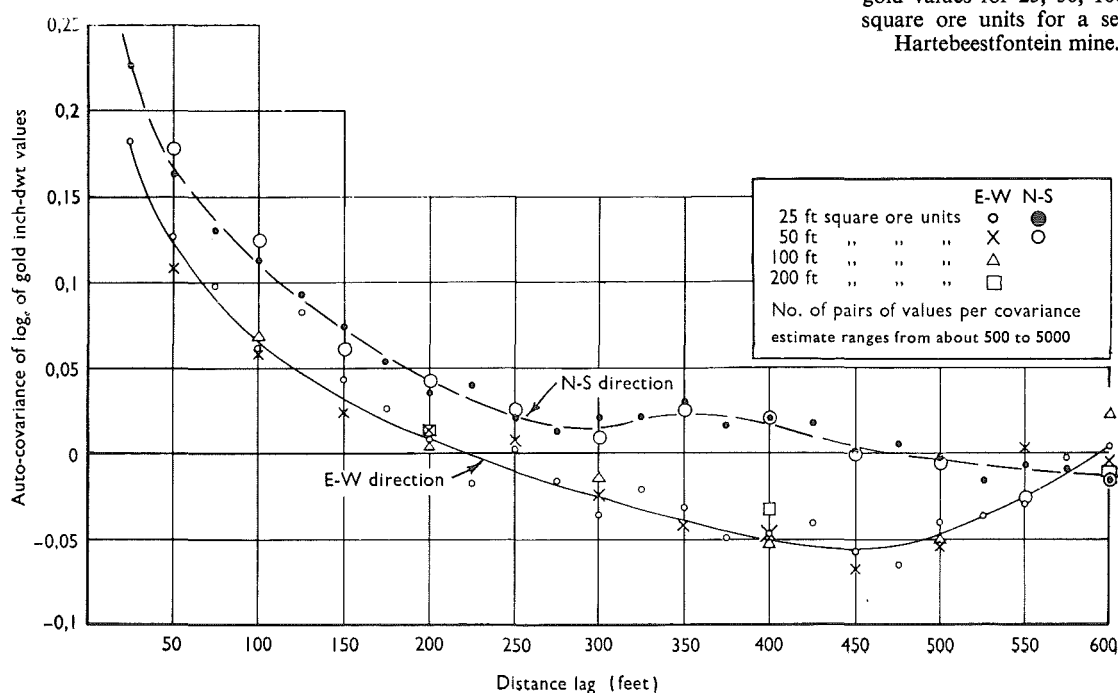
Fig. 18. Illustration of the agreement between auto-covariance patterns of gold values for 25, 50, 100, and 200-ft square ore units for a section of the Hartebeestfontein mine.

2.2 Geostatistical terms and definitions
(Main references: Matheron, 1960, 1971; David, 1977; Rendu, 1978)

The following are some basic terms and definitions (for the *nugget effect*, see §2.3.1):

2.2.1 Regionalised variable
The characteristic of a spatial phenomenon which exhibits a spatial structure is called a regionalised variable. It shows some continuity, cannot be represented by an exact mathematical function of its location, and usually exhibits both a random and structured aspect.

2.2.2 Population, field
As in statistics, the population is the collection of all the possible values of the characteristic concerned within the *domain* or *geometric field* or *universe* in which they occur, which may be an ore body's volume or area.

2.2.3 Support, ore unit
The physical volume (or area), including its orientation and shape, to which a population member relates forms the latter's *geometric support*; for example, the core from a borehole, or the cross-sectional area of a channel sample cut across a bedded deposit. In these notes the geometric support is frequently also referred to as the *ore unit*.

2.2.4 Point values and regularisation
The point value of a characteristic corresponds to an infinitesimally small geometric support. This concept can be useful in the theoretical analysis of spatial structures. In practice this is a purely theoretical concept as characteristics can only be related to physically meaningful supports — volumes (or areas) such as ore samples or ore blocks, and we then deal with *regularised regionalised variables*.

2.2.5 Intrinsic hypothesis — second order stationarity
A regionalised variable obeys the intrinsic hypothesis when the difference between two values of the variable has an expectation and a variance which are both independent of the location of the sampling points within the field and depend only on the vector of displacement between the two values; that is, when the variogram exhibits *stationarity*. The field can then be accepted as *homogeneous*. Second order stationarity applies when a finite population variance and covariances exist and the latter as well as the variable itself have expectations independent of the location within the field.

2.2.6 Semivariogram
The semivariogram function $\gamma(\mathbf{h})$
= half the average of the squares of the differences between pairs of values of the variable concerned at a distance apart measured by the vector $\mathbf{h}$; and accepting $\overline{x(z) = x(z + \mathbf{h})}$, i.e. $\overline{x(z) - x(z + \mathbf{h})} = 0$, 　(21)
= half of the variance of these differences,
= half the *error (or extension) variance* in accepting value $x(z)$ as an estimate of value $x(z + \mathbf{h})$ at a position $\mathbf{h}$ units away from z,

$$= \tfrac{1}{2} E \left[\left\{ x(z) - x(z + \mathbf{h}) \right\}^2 \right] \qquad (22)$$

= $\tfrac{1}{2}\sigma^2$ of difference between $x(z)$ and $x(z + \mathbf{h})$

= $\tfrac{1}{2}\sigma^2_{(x-y)}$,

and is estimated by

$$\frac{1}{2n} \sum_{i=1}^{i=n} \left\{ x(z_i) - x(z_i + \mathbf{h}) \right\}^2 \qquad (23)$$

2.2.7 Average semivariogram value
From classical statistics in dealing with correlated variables x and y:

$$\sigma^2_{(x-y)} = \sigma^2_x + \sigma^2_y - 2\,\mathrm{Cov}_{xy}, \qquad \text{see (13)}$$

and therefore,

$$2\,\gamma(\mathbf{h}) = 2\,\sigma^2_x - 2\,\mathrm{Cov}\,(x_z,\, x_{z+\mathbf{h}}),$$

x and y being replaced by the regionalised variable x_z at locations z and $z + \mathbf{h}$,

$$\gamma(\mathbf{h}) = \sigma^2_x - \mathrm{Cov}\,(x_z,\, x_{z+\mathbf{h}}), \text{ i.e.}$$

$$\left. \begin{array}{l} \text{Semivariogram} \\ \text{value at lag } \mathbf{h} \end{array} \right\} = \left(\begin{array}{c} \text{population} \\ \text{variance} \end{array} \right) - \left(\begin{array}{c} \text{Covariance for pairs} \\ \text{of values } \mathbf{h} \text{ units apart} \end{array} \right). \qquad (24)$$

(See also (13) in Section 1; and David, 1977, §5.3.2)
Note that in valuing parts of the population (or ore body) these equations will be applicable only under conditions of second order stationarity.

In the case where all possible lags $\mathbf{h}$ are taken into account and the results are pooled, we will in effect have a complete and perfect random selection of pairs of values from the population, and from classical statistics the two values within the pairs will then on average be uncorrelated and their covariance will be zero. Hence the average $\gamma(\mathbf{h})$ for all possible values of $\mathbf{h}$, which is the *average semivariogram value* or $\bar{\gamma}(\mathbf{h})$, is given by

$$\sigma^2_x - 0 = \sigma^2_x = \text{population variance.} \qquad (25)$$

2.2.8 Grading

This is a particular case of regularisation where, for example, a three-dimensional ore body such as a bedded deposit with a relatively narrow thickness is analysed on a two-dimensional basis by dealing with the metal contents or *accumulations* within unit *areas* in the general plane of the ore body. This would be grading of the *first order* and the variable would then take the form, for example, of gold g m^{-2}, or cm g/t or inch-dwt/t (see §1.4).

2.2.9 Isotropy, anisotropy

When the semivariogram $\gamma(\mathbf{h})$ is independent of the direction of lag $\mathbf{h}$ — that is, where $\mathbf{h}$ need not be measured as a vector — the variogram is *isotropic*. Where the variogram changes with the direction of lag $\mathbf{h}$, it is *anisotropic*.

2.2.10 Proportional effect

A reasonably common feature in ore bodies is that the value of the semivariogram varies with the average grade of the relevant section of the population studied and a *proportional effect* can then be present. In the case of a lognormally distributed variable, for example, a proportional effect arises from the fact that the population variance as well as the semivariogram is proportional to the square of the average grade. A logarithmic transformation of the variable will eliminate this effect. Suitable adjustments based on the square of the 'local' mean grade have to be made if the analysis is done on untransformed values (David, 1977, pp. 170-4).

2.2.11 Linear equivalent

The length of a linear segment of an ore body, the grade of which has the same variance within that ore body as that of a two- or three-dimensional ore unit, is called the linear

equivalent (λ) of the ore unit. For the de Wijsian model (see §2.3 below) the linear equivalent for a rectangle measuring $(a \times b)$ units can be accepted at $(a + b)$ and thus for a square area of a^2 units as $2a$.

2.2.12 Kriging

Kriging was the name given in 1960 by Matheron to the *multiple regression* procedure for arriving at the *best linear unbiased estimator* or best linear *weighted moving average* estimate of the ore grade for an ore block (of any size) by assigning an optimum set of weights to all the available and relevant data inside and outside the ore block. It has two main advantages, namely the avoidance of systematic bias errors (see §1.17–1.21) and the minimisation of the error of estimation, the *kriging error*.

The estimation and the inversion of the covariance matrix required to establish the optimum weights, form the main parts of any kriging procedure and it is in this respect that a model of the spatial structure of the ore values is not only useful but almost indispensible.

Only simple kriging techniques will be discussed here. The more sophisticated geostatistical procedures such as *universal kriging* and *generalised increments* (Matheron, 1971; Guarascio, 1976) will not be considered and in any case often show little or no advantage over the straightforward approach (David, 1977).

2.3 The lognormal distribution and de Wijsian spatial structure

(Main references: Matheron, 1960, §106; De Wijs, 1951/3)

The following is a synthesis of an ideal process which would produce a lognormal frequency distribution and a de Wijsian spatial structure for ore values. Consider an ore body with a mean metal grade of m, and where the total metal content is introduced unequally between two halves of the ore body, resulting in grades for the two halves of $m(1 + d)$ and $m(1 - d)$ respectively, where $d = de$ *Wijs's coefficient of deviation*. On a second subdivision on the same basis the resulting 4 ore blocks would have grades of

$$m(1 + d)^2; \quad m(1 + d)(1 - d); \quad m(1 - d)(1 + d); \quad m(1 - d)^2.$$

After k subdivisions, there will be 2^k blocks with the following grades:

Blocks:	1 block	k blocks
Grades:	$m(1 + d)^k$	$m(1 + d)^{k-1}(1 - d)$

Blocks:	kCr blocks	1 block
Grades:	... $m(1 + d)^{k-r}(1 - d)^r$ ...	$m(1 - d)^k$

Thus the probability of a block with a grade x equal to $m(1 + d)^{k-r}(1 - d)^r$ will be $kCr/2^k$, which is the probability of drawing tails r times when tossing a coin k times. The variable r is therefore binomially distributed with variance $\frac{1}{4}k$. Also since:

$$\log x = \log m + k \log(1 + d) + r \log \{(1 - d)/(1 + d)\},$$

the logarithm of the grade is a linear function of r. Hence the grade x is logbinomially distributed and *tends to lognormality* for large k, with a logarithmic variance of

$$\sigma^2 = \tfrac{1}{4} k \log^2 \{(1 - d)/(1 + d)\}. \tag{26}$$

If $\quad V =$ volume of original ore body,
and $\quad v =$ volume of each of the 2^k ore blocks,
$\quad\quad V/v =$ number of ore blocks $= 2^k$,
and thus de Wijs's model requires that this variance be pro-

portional to log (V/v), thus

$$\sigma^2 = \alpha \log (V/v) \tag{27}$$

where $\alpha =$ *logarithmic intrinsic or absolute dispersion*,

and $\quad \alpha = (1/4 \log 2) \{\log (1 - d)/(1 + d)\}^2. \tag{28}$

Note that for infinitely small volumes v, or *point samples*, (V/v) as well as σ^2 will become infinitely large and therefore meaningless.

Matheron also showed that if the grades of blocks v are logbinomially distributed, grades for blocks of size $2v$ will also be logbinomially distributed, hence proving the 'permanency' of the logbinomial, and for large k also of the lognormal. He further showed that the same conclusions apply if d is not fixed but is replaced by a random variable with mean d. This '*permanency' of the lognormal pattern*, confirmed by many practical observations (Krige, 1951a), means that where this model applies to the individual sample or borehole values, *the distribution of grades of ore blocks of any specified size can also be accepted as lognormal.*

If V and v are accepted as geometrically similar volumes, then (27) changes as follows:

$$\sigma^2 = \alpha \log (V/v) \quad\quad \text{(for volumes)}$$

$$= (3\alpha/2) \log (A/a) \quad\quad \text{(for areas)}$$

$$= 3\alpha \log \frac{\Lambda}{\lambda}, \tag{29}$$

where A and a are the areas (geometrically similar) of a two-dimensional ore body and its constituent ore blocks respectively, and Λ and λ are the corresponding lengths for a one-dimensional ore body or the linear equivalents for the volumes or areas concerned.

Then $\alpha = (1/6 \log 2) \{\log (1 - d)/(1 + d)\}^2$ for a 2-dimensional body $\tag{30}$

and $\quad \alpha = (1/12 \log 2) \{\log (1 - d)/(1 + d)\}^2$ for a 1-dimensional body $\tag{31}$

Also, since the variance of a lognormally distributed variable

$$= m^2 (e^{\sigma^2} - 1), \quad\quad\quad\quad \text{see (3)}$$

this variance for a two-dimensional ore body can be expressed as

$$= m^2 \left\{ \left(\frac{A}{a}\right)^{3\alpha/2} - 1 \right\}. \tag{32}$$

2.3.1 The nugget effect

Where the regionalised variable concerned represents, for example, the actual grades of very small areas of the ore body and, in the extreme case, the point grades, it can readily be visualised that a minute displacement of a single area (or position of a point grade) can result in a substantial change in actual grade due to the presence or absence of the minute metal particles (or 'nuggets' of, for example, gold). Also, where the grades are obtained from ore samples cut manually underground, repeated sampling in the same groove or even of the same material if it could be replaced *in situ*, could give significantly different grades due to errors of sampling as well as assaying.

The combination of these features produces the result that the observed grade of an ore unit will have an *error variance* or *nugget effect* quantifying the variation which will be observed between the grades found on the theoretically repeated sampling of the same unit. If we are dealing with

larger units where their grades are estimated from one or more small ore samples within each unit volume or area, a further *error* (or *extension*) *variance* is introduced due to the 'extension' of the grade of the sample(s) to that of the unit volume or area, and then the 'nugget effect' for such larger units will be larger and incorporate all the above-mentioned sources of error and variation. Note that with a minute displacement of a substantial area or volume there will be no noticeable change in its *actual* grade and therefore *the nugget effect for the actual grade of an ore block or ore unit of any reasonable size will be zero.*

Equation (29) is applicable to the logarithmic variance of the actual grades of the ore blocks and, where these are estimated, the *logarithmic nugget effect N* (or error variance) therefore has to be added.

To avoid confusion, the variances and semivariograms of *actual* values of regularised variables (excluding the nugget effect) will henceforth, unless evident, be shown in square brackets, for example, $[\sigma^2]$ and $[\gamma(\mathbf{h})]$, and where these refer to *estimates* inclusive of the nugget effect, without brackets. Hence
$$\sigma^2 = N + [\sigma^2]$$
$$\gamma(\mathbf{h}) = N + [\gamma(\mathbf{h})]. \tag{33}$$
The main advantage of de Wijs's model is that it explains and links the lognormal frequency distribution pattern and the de Wijsian spatial structure for the ore values via one basic parameter α (the absolute dispersion) plus the nugget effect N. From the above and (54)–(57) below it will be evident that all logarithmic variances, covariances and semivariogram values can be expressed in terms of α and N for populations emanating from any size of geometric field V or A — the volume or area of the ore body—and for any size of geometric support v or a — the volume or area of ore units.

2.3.2 *Additivity of variances*
A further feature to be recorded and not confined to the de Wijsian model only, is the *additivity of variances* (Krige, 1951a). If v and v' are subvolumes within the ore body with volume V, and $v' > v$, then from classical statistics:

Population variance = average variance within strata plus variance between strata.
Thus, variance of grade of v within V is given by variance of v within v' + variance of v' within V,

and $\qquad [\sigma_v^2/V] = [\sigma_v^2/v'] + [\sigma_{v'}^2/V]$

or $\quad \sigma_v^2/V - N_v = \sigma_v^2/v' - N_v + [\sigma_{v'}^2/V]$,

thus $\qquad \sigma_v^2/V = \sigma_v^2/v' + [\sigma_{v'}^2/V]$.

Similarly, for areas a, a' and A,

$$[\sigma_a^2/A] = [\sigma_a^2/a'] + [\sigma_{a'}^2/A]$$

and $\quad \sigma_a^2/A = \sigma_a^2/a' + [\sigma_{a'}^2/A] \tag{34}$

$$= \sigma_a^2/a' + \sigma_{a'}^2/A - N_{a'}.$$

In the lognormal case, (34) will also effectively apply to the logarithmic variances of the grades for these areas (see (41) and §3.3.3).

2.4 Features of the nugget effect: error or extension variances
(References: Matheron, 1960, Sect. 4/5; Matheron 1971; David, 1977; Rendu, 1978)
Let $\quad a$ = the size of unit areas on a regular grid within the field,

A = population area,
σ_a^2 = variance of grade estimates of unit areas a (based on one ore sample at centre of each subarea) within population A,
$[\sigma_z^2]$ = variance of actual grades of unit areas a within A,
σ_{az} = covariance of grade estimates a and actual grades z of the unit areas within A.

Then *error variance* σ_{Ea}^2 of grade estimates for unit area a

= variance of (grade estimate − actual grade)
= *extension variance* of ore sample grade to unit area a
= nugget effect N_a
$$= \sigma_a^2 - 2\sigma_{az} + [\sigma_z^2].$$

For the lognormal–de Wijsian model and dealing with the logarithms of the values:

$$\sigma_{az} = [\sigma_z^2] + 3\alpha/4$$

if the individual ore samples are taken at the centres of the unit areas (Matheron, 1960),
and the logarithmic error variance becomes:

$$\sigma_{Ea}^2 = N_a = \sigma_a^2 - [\sigma_z^2] - 3\alpha/2. \tag{35}$$

Also, the error variance of the mean of all the grade estimates for unit or subareas a as an estimate of the mean of the A population (all subareas a sampled and error variances for subareas assumed to be uncorrelated) is given by

$$\sigma_{EA}^2 = \sigma_{Ea}^2/\text{no. of subareas} = N_a/(A/a) = aN_a/A$$
(for any spatial structure).

For all subareas sampled at their centres, and where the logarithmic-de Wijsian model applies, the error variance for the mean of the logs will be:

$$\sigma_{EA}^2 = N_A = \frac{a(\sigma_a^2 - [\sigma_z^2] - 3\alpha/2)}{A}. \tag{36}$$

Note that the term $(-3\alpha/2)$ defines the reduction in the logarithmic error variance owing to the samples being taken on a regular grid at the centre of each subarea and not at random. Thus, this reduction is due to substituting systematic for random sampling.

If taken at random within each subarea (for any spatial structure),

$$\sigma_{EA}^2 = N_A = \frac{a(\sigma_a^2 - [\sigma_z^2])}{A}$$
$$= \frac{\text{Error variance within subareas}}{\text{no. of subareas}}, \tag{37}$$

which is the standard statistical formula for stratified random sampling.

If the samples are taken at random over the whole population area,

$$\sigma_{EA}^2 = N_A = \sigma_a^2/\text{no. of samples (i.e. is equivalent to } \frac{a\sigma_a^2}{A}),$$

and (37) thus changes to the standard formula for random sampling.

If all the subareas within the population are not sampled, and the average fraction sampled is f (spread at random), then, as a close approximation, the effective size of each sampled subarea a could be accepted at a/f, and then for the logarithmic model:

σ_a^2 remains unchanged but $[\sigma_z^2]$ is reduced (as $a/f > a$) by

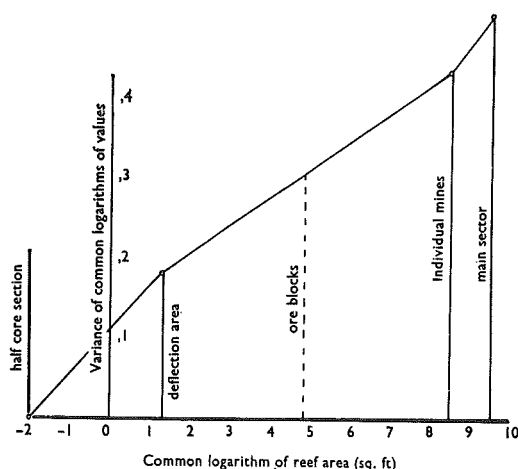an amount = $(3\alpha/2)$ (difference between logs of a/f and a)

Fig. 19. Illustration of the variance-size of area relationship as observed in 1952 for Orange Free State borehole values.

$$= (3\alpha/2) \log \{(a/f)/a\} \qquad \text{see (29)}$$
$$= (3\alpha/2) \log (1/f).$$

Then $\qquad \sigma_{Ea}^2 = N_a = \sigma_a^2 - 2\,\sigma_{az} + [\sigma_z^2] - (3\alpha/2)\log(1/f)$

and $\qquad \sigma_{az} = [\sigma_z^2] - 3\alpha/2 \log(1/f) + 3\alpha/4,$

therefore $\quad \sigma_{Ea}^2 = N_a = \sigma_a^2 - [\sigma_z^2] - (3\alpha/2)$
$$+ (3\alpha/2)\log(1/f) \qquad (38)$$

Also, $\qquad N_A = \dfrac{a(N_a)}{Af}, \qquad\qquad (39)$

where $\qquad N_a = \sigma_a^2 - [\sigma_z^2] - 3\alpha/2 + (3\alpha/2)\log(1/f)$

for samples at centres of subareas a,

$$\text{or} = \sigma_a^2 - [\sigma_z^2] + (3\alpha/2)\log(1/f)$$

for samples at random within the subareas a.

Where f is reasonably close to 1, the factor $(3\alpha/2)\log(1/f)$ can be ignored; also, if the data density within field A is very variable, the use of the average f value in the above formulae could lead to biased results. *For lognormally distributed variables the application of (39) to the logarithmic*

nugget effect is not advisable and the alternative approach in (41)—(§2.5)—should be used.

Where an ore body or part thereof has been sampled on some other reasonably systematic pattern, for example at regular intervals along drives at various elevations, estimates of the error or extension variances can be obtained for the de Wijsian case (and for certain other standard models such as the *spherical variogram*) using close graphical approximations (David, 1977, §8.3).

It must be stressed again that all the logarithmic error variances covered above will have corresponding *skew limits of error* (see §1.10 and Table 4) with, as a first approximation, an assumption of lognormality of the error distribution.

For the lognormal-de Wijsian case N_a is the *logarithmic nugget effect* and is usually estimated by extrapolation of either

(i) the semivariogram to a lag of log $(0,223\lambda)$, or log $(0,446)$ for unit square areas (see §2.6), or

(ii) the variance-size of area relationship (see §2.5) to unit area a.

2.5 The de Wijsian variance-size of area relationship
(References: Krige, 1951a, 1952, 1966b; Matheron, 1960,1971; De Wijs, 1951/3)

The fact that the logarithmic variance of gold values shows a rising linear trend when plotted against the logs of the population areas concerned was observed by the author during the initial studies of underground gold values and of the Orange Free State borehole values (Krige, 1951a, 1952; Fig. 19) and was linked with the de Wijsian model in a regional study (Krige, 1966b) as shown in Fig. 20. The mathematical relationships concerned can be summarised as follows:

Let $\quad \sigma_{a/A}^2 = $ logarithmic variance of grade estimates of subareas a within population area A, *all areas being of similar shape and orientation.*

$\quad N_a = $ logarithmic nugget effect for subareas a, i.e. logarithmic error variance of the grade estimates of a.

Then, from (29)

$$\sigma_{a/A}^2 = N_a + (3\alpha/2)\log(A/a), \qquad (40a)$$

that is, for varying population sizes A and fixed unit area a, the variance-size of area (log) relationship is linear and *the*
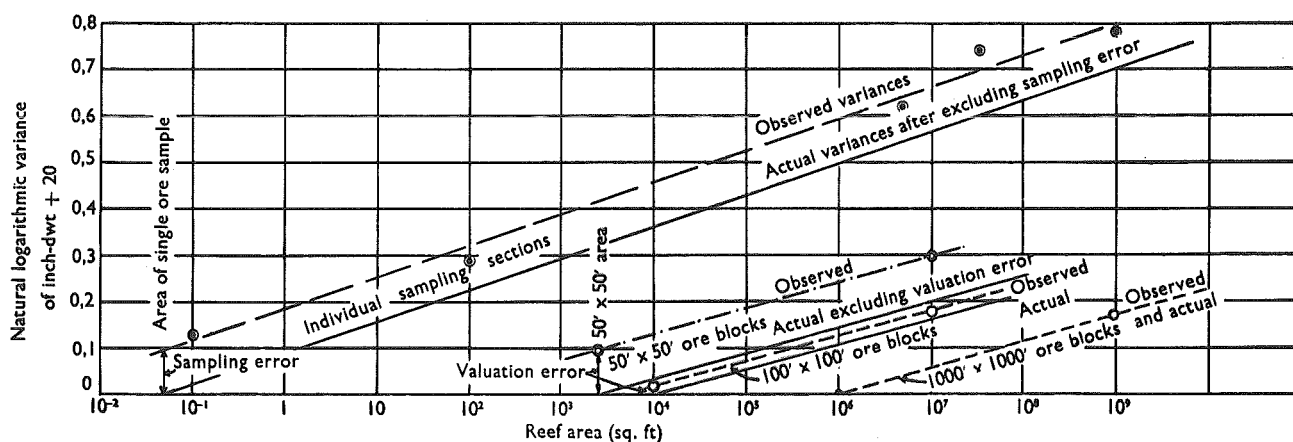

Fig. 20. Illustration of the variance-size of area relationships observed in 1966 for gold values in the Klerksdorp goldfield.

nugget effect N_a *can be estimated by extrapolation of this linear trend to a population area* A *of size* a.

If nugget effect $= 0$, which means the actual grades of the subareas are known, then as in (29) above,

$$[\sigma_{\bar{a}/A}^2] = (3a/2) \log (A/a). \qquad (40b)$$

Note that unless otherwise indicated, all population and error variances σ^2, when used for lognormally distributed variables, will refer to the *logarithmic* variances, the variances of the logarithmically transformed values. Similarly, semivariogram values $\gamma(\mathbf{h})$ and covariances will be based on these transformed values. A practical demonstration of the fact that the log variance or relative variability for the grades of unit areas a within population A will be similar if the ratio A/a is the same, regardless of the absolute sizes of A and a, is provided by Fig. 21.

For two sizes of subareas a_1 and a_2 ($a_1 > a_2$):

$$\sigma_{\bar{a}_2/A}^2 - \sigma_{\bar{a}_1/A}^2 = N_{a_2} + (3a/2) \log (A/a_2) - N_{a_1}$$

$$- (3a/2) \log (A/a_1)$$

$$= N_{a_2} - N_{a_1} + (3a/2) \log (a_1/a_2) \qquad (41)$$

$$= N_{a_2} - N_{a_1} + 3a \log (\lambda_1/\lambda_2) \qquad (41a)$$

$$= N_{a_2} - N_{a_1} + [\sigma_{\bar{a}_2/a_1}^2] \qquad (41b)$$

$$= \sigma_{\bar{a}_2/a_1}^2 - N_{a_1} \qquad \text{(see (33))} \quad (41c)$$

These features are shown graphically in Fig. 22.

2.5.1 *Block variances for selective mining estimates*

Selective mining cannot be done on the actual block values with variance as in (40b) because these are never known in advance, and can therefore only be based on imperfect *estimates* of the actual block values. It should be stressed that the variance $\sigma_{\bar{u}/A}^2$ in (40a) in turn corresponds to the grade estimates of the a areas before applying any regression adjustments (see §1.20) or geostatistical 'kriging' procedures (see §2.11). This corresponds in Fig. 15 to the variance along the x-axis. The variance along the y-axis, that is, of the actual values for the ore blocks, will be exclusive of the nugget effect, as in (40b):

$$(3a/2) \log (A/a). \qquad (42)$$

The variance of the regressed block values will under certain conditions be given by:

$$\frac{(\text{Variance of actual block values})^2}{\text{Variance of unregressed block valuations}} \quad \text{(see (20))}$$

$$= (42)^2/(40a) \qquad (42a)$$

or, more generally and preferably, (see (18a), (19a) and (20a))

= Variance of actual block values (42) − error variance of regressed values. (42b)

In the lognormal modelling of an ore body for a selective mining process, the following will therefore apply (see also §1.20):

Where regression is expected to be used in arriving at the block valuations during mining operations, (42a) or preferably (42b) should be used in estimating the block variance for predicting the payable ore reserves — that is, the percentage payability and pay value from (6) and (7) or graphically from Fig. 4. If regression is not applied and (40a) is used for pre-
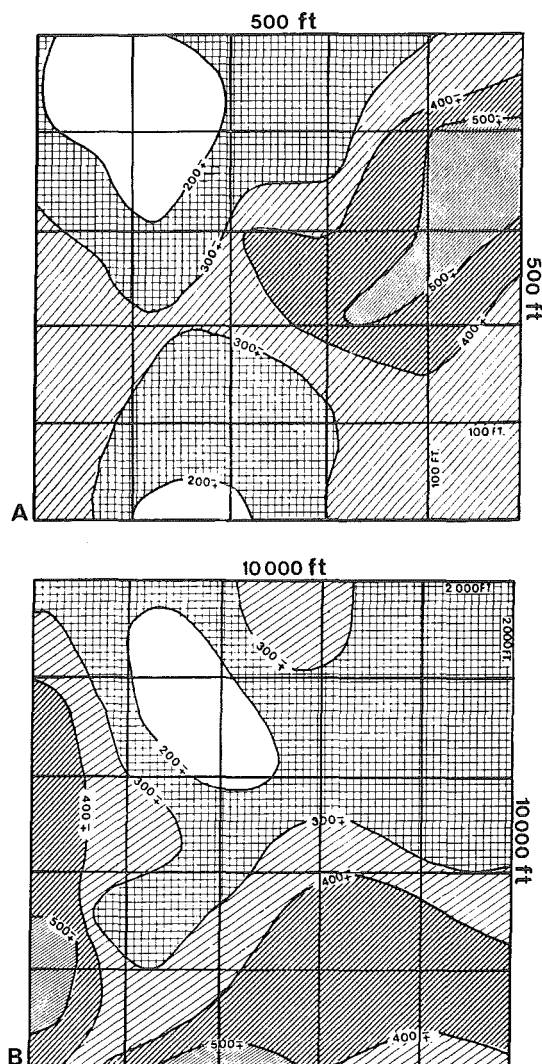

Fig. 21. Typical gold inch-dwt trend surfaces in the Klerksdorp goldfield on the basis of two-dimensional moving averages for two sections with similar average grades (1966). (*a*) Moving averages of 100 × 100 ft areas within a mined-out section of 500 × 500 ft. (*b*) Moving averages of 2000 × 2000 ft areas within a mined-out section of 10 000 × 10 000 ft.
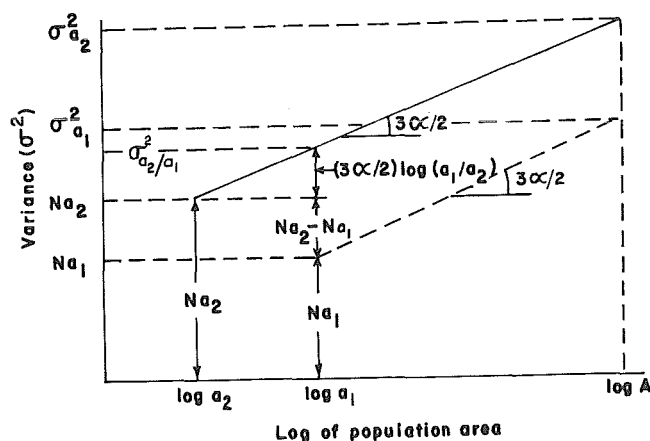

Fig. 22. Graphical illustration of the de Wijsian-size of area relationship.

diction, the pay value and the payability will both be seriously biased (see §1.17–1.20).

Where the more sophisticated kriging procedures (see §2.11) are to be used for block valuations, (42b) will still apply but the error variance of the regressed values will be replaced by the somewhat smaller kriging variance of the block valuations (see §2.11) and the selection process will then be more efficient; (42b) will be correct if the mean grade of the ore body is known and a very close approximation where this is not the case (David, 1977, p. 256).

Where the lognormal model is not applicable, methods such as *transfer functions* and *disjunctive kriging* (Guarascio, 1976; Kim, 1977; Marechal, 1977) can be used for effectively estimating the distribution of the relevant block grades. However, the flexibility and advantages of the lognormal-de Wijsian model are such that consideration should only be given to another model after tests have shown conclusively that the basic values cannot be normalised for any level of the additive (third) parameter.

2.5.2 *Outliers in deflection values*

Where the average logarithmic variance within deflection areas can be estimated directly from the data on the property, or from a variance-size of area graph, or be inferred from similar neighbouring properties, any of the recognised procedures can be used (based on normal theory for the transformed values) to test for outliers, which are any 'freak' values arising from core losses or contamination. These procedures will strictly be applicable only if the logarithmic variance within deflection areas can be accepted as stationary and if the value distributions within such areas are truly lognormal.

2.6 The de Wijsian semivariogram

(References: Matheron, 1960, 1971; David, 1977; Rendu, 1978)

If a regularised regionalised variable follows the de Wijsian model with a logarithmic intrinsic absolute dispersion of α (see §2.3), Matheron has shown that for $\mathbf{h} \gg \lambda$:

$$\gamma(\mathbf{h}) \simeq N + 3\alpha \left(\log \frac{\mathbf{h}}{\lambda} + \frac{3}{2}\right)$$

$$\simeq N + 3\alpha \log (\mathbf{h}/0{,}223\lambda) \qquad (43a)$$

where N = nugget effect (logarithmic),

α = absolute dispersion (logarithmic),

$\mathbf{h}$ = lag between centres of unit areas,

λ = linear equivalent of the unit area (or support),

and $\gamma(\mathbf{h})$ = semivariogram value of logs of observed values.

Also, $[\gamma(\mathbf{h})]$ = semivariogram value of logs of actual values with no nugget effect

$$\simeq 3\,\alpha \log (\mathbf{h}/0{,}223\lambda). \qquad (43b)$$

The logarithmic–de Wijsian semivariogram can therefore be accepted as linear when plotted against log $\mathbf{h}$.

For the case where the unit areas (the supports) measure unit length squared, and $\mathbf{h}$ is measured in the same units:

$\lambda = 2$,

and $\gamma(\mathbf{h}) \simeq N + 3\alpha \log (\mathbf{h}/0{,}446)$. $\qquad (44)$

For $\mathbf{h} \to 0$, $\gamma(\mathbf{h}) \to -\infty$; $\mathbf{h} \to \infty$, $\gamma(\mathbf{h}) \to \infty$,

and for $\lambda \to 0$ (i.e. point level), $\gamma(\mathbf{h}) \to \infty$.

This model cannot, therefore, be used at the point level but only for regularised variables. Furthermore, it is not bounded and tends to infinity as $\mathbf{h}$ increases, thus suggesting that second order stationarity cannot apply. However, in practice we are
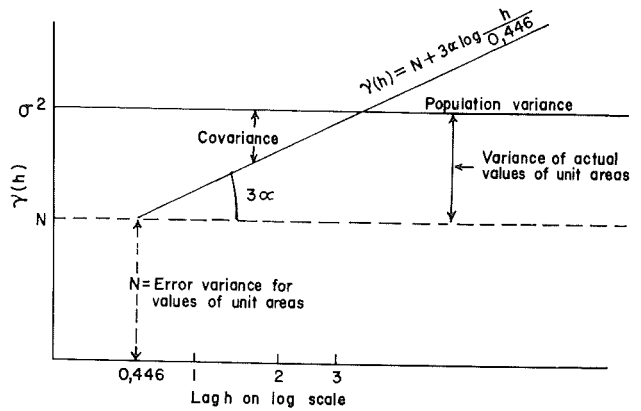

Fig. 23. Illustration of the logarithmic-de Wijsian semi-variogram model.

usually dealing with a finite geometric field (or defined ore body or section thereof) and in that case we have a finite population variance; also, $\mathbf{h}$ cannot exceed the maximum linear measurement across the field, and $\gamma(\mathbf{h})$ will, therefore, also be bounded.

Figure 23 shows the logarithmic–de Wijsian semivariogram as a straight line when plotted against the logarithm of lag and with a slope $= 3\,\alpha$. Also shown is its relationship with the covariance and nugget effect.

Although (43) and (44) are not applicable for $\mathbf{h} < \lambda$, the theoretical position on this model for $\gamma(\mathbf{h}) = N$ is interesting and useful in practice.

For $\gamma(\mathbf{h}) = N$, $\log \dfrac{\mathbf{h}}{\lambda} = -\dfrac{3}{2}$

and for square unit areas with sides of unit length ($\lambda = 2$),

$$\log \mathbf{h} = -\frac{3}{2} + \log 2$$

$$\mathbf{h} = 0{,}446 = 0{,}223\lambda. \qquad (45)$$

Thus, the nugget effect can be estimated by extrapolation of the linear semivariogram down to a lag of 0,446 units.

2.6.1 *Mean of semivariogram within field*

It can be shown from (25) and (40) that the mean of the semivariogram within the whole field A, $\bar{\gamma}(\mathbf{h})$, or the population variance, equals the semivariogram value in (43a) for a lag $\mathbf{h}$ corresponding to:

$$\log (\mathbf{h}/\lambda) + \frac{3}{2} = \frac{1}{2} \log (A/a);$$

but $\lambda = 2$ and $a = 1$ for square unit support areas.

Thus $\log \mathbf{h} = \log (0{,}223\,A) = \log (0{,}446\,H)$ $\qquad (46)$

$\mathbf{h} = 0{,}446\,H$ $\qquad (47)$

$= 0{,}223\,A,$ $\qquad (47a)$

where A = linear equivalent for the whole field, and

H = length of the side of a square field.

Equations (47) and (47a) hold irrespective of the size of the unit areas a relative to the field A, and of the corresponding nugget effect N_a for these unit areas; (47a) also holds for any shape of population area.

This feature can be demonstrated in a practical way by studying the frequency distribution of the lags between pairs of values for a very small size of support over the complete range of the variogram, and for fields varying from the linear

to a rectangle and to a square area, all with the same linear equivalent Λ. If all possible pairs of values for all lags and all directions are taken, they will be equivalent to a complete set of statistically random samples of two from a population, and then

$$\bar{\gamma}(h) = \tfrac{1}{2}\,(2\,\sigma^2 - 2\,\mathrm{Cov})$$

$$= \sigma^2, \tag{48}$$

which is the population variance (see (13) and (25)).

Figure 24 shows the shape of a typical logarithmic–de Wijsian semivariogram (assumed isotropic) plotted against an arithmetic lag scale on the x-axis, with below this scale the nature of the observed lag frequency distributions for the three populations in (i) a linear field of 200 lags, (ii) a rectangular field of 160×40 lags, and (iii) a square field of 100×100 lags, the latter two both having a linear equivalent of 200 lags (see §2.2.11). The means of the semivariogram values over the ranges of lags, weighted by the lag frequencies in each of the three cases, were found to be very close to 0,535 (the population variance), and correspond in each case to a lag of about 45 units, which was $22\tfrac{1}{2}\%$ of the linear equivalent of 200, as expected from (47a).

Note that as the average semivariogram value within a defined field must equal the population variance, a semivariogram starting at values *below the average* (as it invariably does unless a spatial structure is not present) must also cross the average line and reflect some values *above the average*. In practical situations this latter effect usually occurs at lags greater than those used for kriging estimates. The effect can therefore usually be disregarded, as is effectively done in using semivariogram models such as the *spherical* with a maximum value or sill equal to the population variance (David, 1977).

2.7 The de Wijsian semivariogram and covariances for different sizes of unit areas (or support) and of geometric fields (population areas)

2.7.1 Variations in size of unit areas

Let $\gamma a_1(h) =$ semivariogram for a_1 units after logarithmic transformation,

$$= [\gamma a_1(h)] + N_1,$$

$\gamma a_2(h) =$ semivariogram for a_2 units $(a_1 > a_2)$,

$$= [\gamma a_2(h)] + N_2,$$

$[\overline{\gamma a_2/a_1}] =$ average semivariogram for a_2 units within a_1 areas (actual),

$\phantom{[\overline{\gamma a_2/a_1}]} =$ average variance of actual values of a_2 units within a_1 areas (exclusive of the nugget effect)   (see (25))

$$= [\sigma^2_{a_2}/a_1] = \sigma^2_{a_2}/a_1 - N_2 \qquad \text{(see (33))}$$

λ_1 and $\lambda_2 =$ linear equivalents of a_1 and a_2 respectively,

N_2 and $N_1 =$ nugget effects for a_2 and a_1 units respectively, and

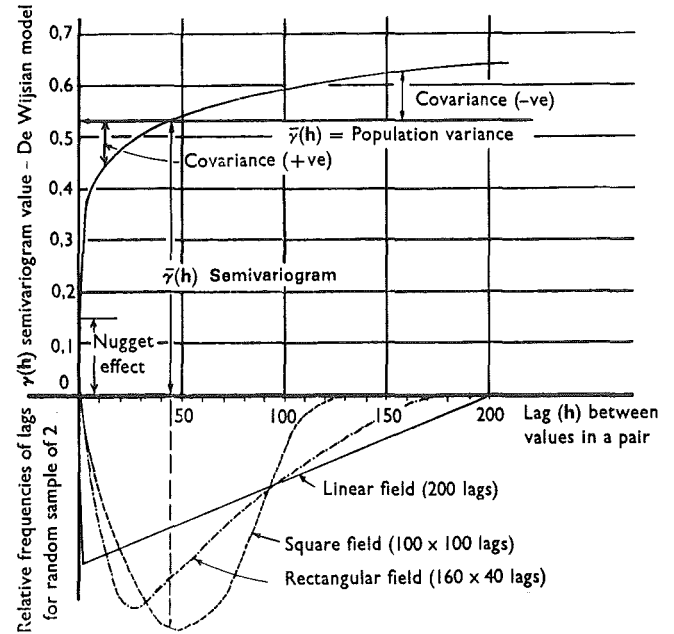$\sigma^2_{a_2}$ and $\sigma^2_{a_1} =$ log variances for the a_2 and a_1 units respectively.

Fig. 24. Graph to illustrate the lag at which the mean value of the semivariogram occurs for the logarithmic-de Wijsian model.

Then, for $h > \lambda_1$:

$$[\gamma a_1(h)] \simeq [\gamma a_2(h)] - [\overline{\gamma a_2/a_1}] \qquad \text{(Rendu, 1978)}$$

and $\qquad \gamma a_1(h) - N_1 \simeq \gamma a_2(h) - N_2 - \sigma^2_{a_2}/a_1 + N_2$

$$\gamma a_1(h) \simeq \gamma a_2(h) - \sigma^2_{a_2} + \sigma^2_{a_1} \qquad \text{(see (41c)) (49)}$$

Also, from (41a),

$$N_2 = N_1 + \sigma^2_{a_2} - \sigma^2_{a_1} - 3\alpha \log (\lambda_1/\lambda_2). \tag{50}$$

These relationships are demonstrated in Fig. 25.

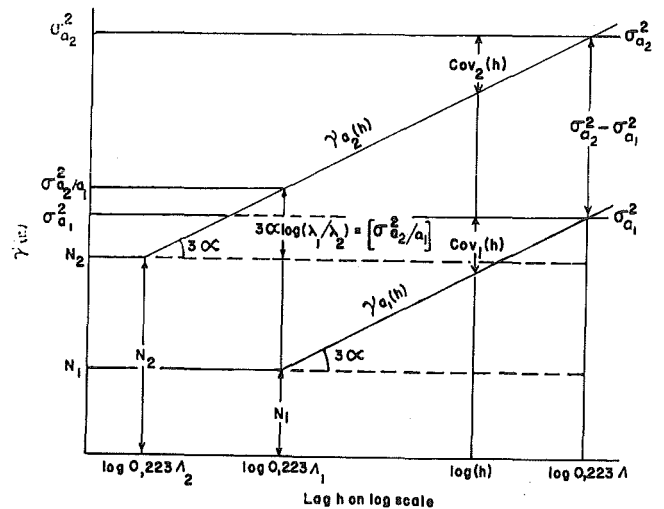Formulae (41) and (50) are applicable provided the variance-size of area graph as well as the semivariogram

Fig. 25. The effect of changes in size of ore units on semivariogram, nugget effect and covariance.

graph for the a_1 and a_2 unit areas are parallel, with a vertical displacement between them of $(\sigma_{a_2}^2 - \sigma_{a_1}^2)$ and with slopes of $3\alpha/2$ and 3α respectively. They have been found to be valid in practical cases even where the graphs deviate significantly from one side to the other of the straight line de Wijsian model for certain area and lag intervals. In such cases the observed curves remained parallel and separated by the same vertical displacement; the slopes $3\alpha/2$ or 3α used in the formulae, as well as the nugget effect, should then be estimated from the smaller fields and the shorter lags, which will be relevant for any kriging applications (see §2.10–2.12 and 3.3).

It should also be noted from (24) and (49) and Fig. 25 that:

Cov$_1$(h) = covariance between a_1 units for lag $\mathbf{h} \gg \lambda_1$

$$= \sigma_{a_1}^2 - \gamma_{a_1}(\mathbf{h})$$

$$\simeq \sigma_{a_1}^2 - \{\gamma_{a_2}(\mathbf{h}) - \sigma_{a_2}^2 + \sigma_{a_1}^2\}$$

$$\simeq \sigma_{a_2}^2 - \gamma_{a_2}(\mathbf{h})$$

$$\simeq \text{Cov}_2(\mathbf{h}), \tag{51a}$$

which is the covariance between a_2 units for lag $\mathbf{h}$, confirming that *the covariances between the values for units of different sizes and with different nugget effects are the same for the same lag* $\mathbf{h}$ (provided $\mathbf{h} \gg \lambda_1$ and the effects of any anisotropy are not significant — see §2.8.3). A practical confirmation of this conclusion for the lognormal case will be evident from a study of Fig. 18. This is a useful feature in setting up the matrices for the kriging solutions. Also, in practice the restriction of $\mathbf{h} \gg \lambda_1$ can often be disregarded (see §2.8.3).

The above conclusions regarding the covariances corresponding to units or supports of different sizes can also be substantiated from the fact that the covariance between the grades of the larger a_1 units of any size for lag $\mathbf{h}$ is the average covariance for all possible pairs of point grades — or of grades of the smaller a_2 units — with one point grade of each pair in the one a_1 unit, and the other point grade in the other a_1 unit (Matheron, 1960; Rendu, 1978). Therefore, if the variogram and covariogram are both linear or almost linear over the range of lag distances between such pairs of point grades (within 2 areas a with centres lag $\mathbf{h}$ apart) and the average of these lag distances closely approximates to lag $\mathbf{h}$, the covariances for different size a_i units or *mixtures of such units* can all be accepted as the same for any specified lag, provided the lag is not short relative to the a_i sizes (see also (56) and (57)).

Thus the covariance between the grades for units a_1 and a_2 at lag $\mathbf{h}$ (for $\mathbf{h} > \lambda_1 > \lambda_2$):

$$= \text{Cov}_{1,2}(\mathbf{h}) \simeq \text{Cov}_1(\mathbf{h}) \tag{51b}$$

$$\simeq \text{Cov}_2(\mathbf{h}).$$

Or Cov$_{i,j}$(h) $\simeq$ Cov$_i$(h) $\simeq$ Cov$_j$(h)

provided $\mathbf{h} > \lambda_i$ (or λ_j) and where i and j can correspond to any size of ore unit a_i or a_j.

Therefore, γa_i (h) $\simeq \sigma_{a_i}^2 - \text{Cov}_j$ (h). $\tag{51c}$
Equations (51) can be of particular use in cases where the available data relate to ore units (or supports) of varying size, for example, variable borehole core lengths. Subject to the above-mentioned conditions and provided the population variances and means (of logs if lognormal) for the different support sizes are estimated on a reasonable basis and there is no significant correlation between grade and support size, the covariance pattern or covariogram against lag — Cov (h) — can in such a case be established using the mixture of all
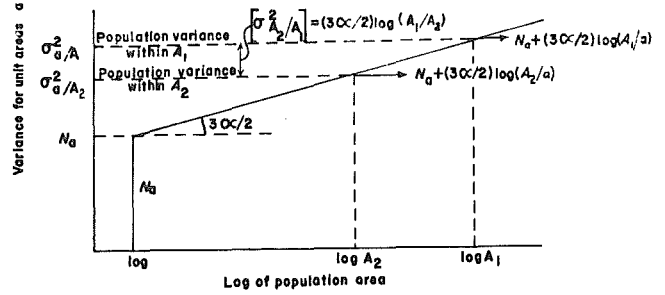

Fig. 26. The effect on variance-size of area relationship of an increase in the size of the population area.

available data. This will enable the variograms for different support sizes to be estimated indirectly from the covariogram and (51c). Note that in the lognormal case, the means of the populations of log values will differ for the common population mean of the untransformed values; the former should be used in the covariance calculations.

2.7.2 Variations in size of field

With the unit areas (support) unchanged, an increase in the size of the geometric field (population area) will result in an increased population variance $\sigma_{a/A}^2$ (see §2.5).

Let $A_1 =$ size of field 1,

$\qquad A_2 =$ size of field 2 $(A_1 > A_2)$.

$\qquad \sigma_{a/A_1}^2 =$ variance of values of a units within A_1,

$\qquad \sigma_{a/A_2}^2 =$ variance of values of a units within A_2,

then $\sigma_{a/A_2}^2 = N_a + (3\alpha/2) \log (A_2/a)$ see (40a)

$\qquad \sigma_{a/A_1}^2 = N_a + (3\alpha/2) \log (A_1/a)$

$\qquad \sigma_{a/A_1}^2 - \sigma_{a/A_2}^2 = [\sigma_{A_2/A_1}^2]$ see (34)

$\qquad\qquad\qquad = (3\alpha/2) \log (A_1/A_2). \tag{52}$

If the field, on extension from size A_2 to size A_1, remains homogeneous, that is, with stationarity of the semivariogram, the latter will not change in definition — see (43) and (44).
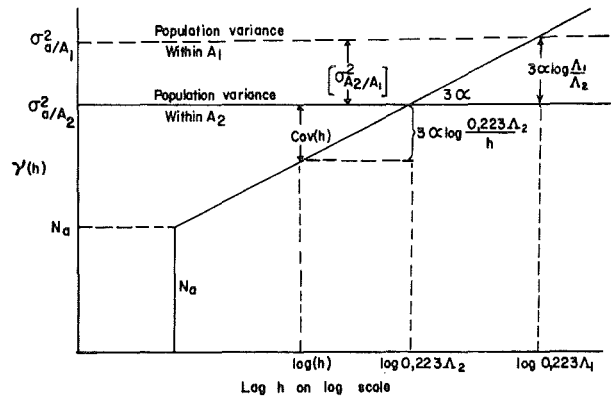

Fig. 27 The effect on the semivariogram of change in the size of population area (from A_2 to A_1).

Thus, $\gamma(h) \simeq N_a + 3\alpha \log (h/0{,}223\lambda)$ (53a)

as N_a, α and λ remain unchanged.

However, as the population variance will increase, the *average of the semivariogram* within the field (as well as the *covariance for any specified lag* h) will be increased to the same extent — see (48) and (52) — by an amount given by

$$(3\alpha/2) \log (A_1/A_2) = 3\alpha \log (A_1/A_2) = [\sigma^2_{A_2}/A_1].$$ (53b)

These effects are shown graphically in Figs 26 and 27. Note that, as indicated by (47), the population variances σ^2_{a}/A_1 and σ^2_{a}/A_2, (the averages of the semivariograms) occur at lags of $0{,}223\,A_1$ and $0{,}223\,A_2$ respectively.

2.7.3 Variations in sizes of support and of field

From (40), (41) and (53a):

$$\sigma^2_{a_2}/A_2 = N_{a_2} + (3\alpha/2) \log \frac{A_2}{a_2}$$

$$= N_{a_1} + \sigma^2_{a_2}/A_1 - \sigma^2_{a_1}/A_1 + (3\alpha/2) \log (A_2/a_1)$$ (54)

and

$$\gamma_{a_2}(h) = N_{a_1} + \sigma^2_{a_2}/A_1 - \sigma^2_{a_1}/A_1 + 3\alpha \log (h/0{,}223\lambda_1).$$ (55)

The covariance for square unit areas a_n at lag h $(h \gg A_n)$

$$= \mathrm{Cov}_{a_n}(h) = (54) - (55)$$ (see (24))

$$\simeq (3\alpha/2) \log (0{,}199\, A_m/h^2),$$ (56)

$$\simeq 3\alpha \log (0{,}223\, \lambda_m/h).$$ (57)

Thus, the covariance between ore values for areas at a lag of h units apart, *i.e. the covariogram, is independent of the size of the support or nugget effect* (see also (51)) *and depends only on the size of the population area* A_m (*or its linear equivalent* A_m), *the absolute dispersion* α, *and the value of* h. Formulae (56) and (57) hold provided h is greater than the linear equivalent λ_n of support or unit area, and provided the effect of any anisotropy can be ignored in applying the formulae (see §2.9 and Krige, 1968). Note that where the observed semivariogram departs from the strict linear model for lags around A_m, (56) and (57) can give misleading results and the covariance should then rather be determined from (24) using the observed population variance and the corresponding semivariogram function as fitted up to the maximum lag of interest for kriging purposes.

2.8 The standard data pattern for gold ore reserves

(References: Krige, 1966; Rendu, 1978)

Early research on the gold values in South African gold mines indicated the need for taking into account available values up to some 400 ft (120 m) away from the centre of a standard area of 100 ft $\times$ 100 ft (900 m²) to be valued. Individual underground sample values along advancing stope faces form an irregular two-dimensional pattern which, over the suggested data-source area of 800 ft $\times$ 800 ft, could cover on average some 400 to 500 values. Furthermore, to do the complete valuation of the scattered ore reserve blocks for a whole gold mine could involve the valuation of some 25 000 to 30 000 standard 100 ft $\times$ 100 ft areas, and the handling of up to 500 individual sample values for each of the areas. This is obviously impractical, and the data were, therefore, condensed or regularised via basic areas of 25 ft $\times$ 25 ft successively into areas of 50, 100 and 200-ft square, with an average of the smaller constituent data squares in each square accepted as the data value for that square. The regularised data pattern used for the valuation of a standard
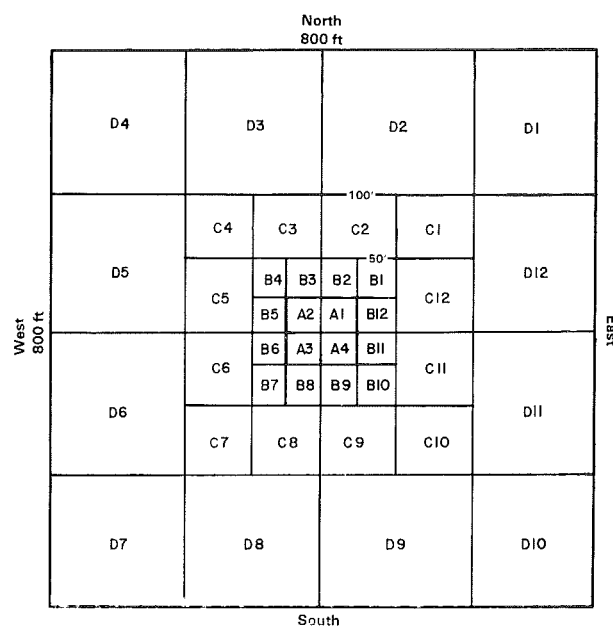

Fig. 28.   The standard data pattern used in gold ore reserve valuations.

100 ft $\times$ 100 ft area is shown in Fig. 28. For practical reasons this procedure was not metricated for mines already using the system when metrication was introduced. Note that the lengths of the sides of the squares in successive peripheries increase logarithmically in conjunction with the log-lag scale for the linear de Wijsian semivariogram. Multiple regression based on such a system of data regularisation is called *random kriging* (David, 1977); it is particularly useful in cases of high variability and where the data which can effectively be used for each block estimate are substantial and have to be condensed to manageable proportions.

For the purpose of calculating the kriged values (or weighted moving averages) for the central standard 100-ft square blocks, the optimum weights to be used for the internal and/or surrounding data squares involve

(i) *the establishment of the complete matrix of covariances* between all the data squares, and between them and the central square to be valued; it is for this purpose that a complete semivariogram model is required, thus avoiding the substantial effort involved in estimating without a model all the covariances directly from the basic data in mined-out areas, and

(ii) *the inversion of this matrix.*

Even on this condensed data pattern this complete procedure is still not a practical proposition for each of the 25 000 to 30 000 100-ft blocks to be valued, and a standard weighting table is therefore used; it is established from a compromise between the optimum weights for a series of the more common data patterns actually found in practice (see §3.3.5.3).

As shown earlier (§2.7.1), the covariances between data blocks are normally not affected by their sizes, but the variances (observed and actual after elimination of the nugget effect) are affected and it is useful to establish the relevant relationships. These have to allow for the fact that, owing to the irregular grid of sampling values, a percentage of 25-ft squares will be blank when their values are averaged in sets of 4 to arrive at the values of the 50-ft data squares, and the same will apply to the 50-ft squares used for establishing the values for the 100-ft data squares and to the 100-ft squares in arriving at the values of the 200-ft data squares.

Let A = population area measured in 25-ft square units.

n_{25} = number of   25 ft sq. unit areas with values.

n_{50} =   „   50   „   „   „   „   „

n_{100} =   „   100   „   „   „   „   „

n_{200} =   „   200   „   „   „   „   „

Then $f_{25} = \dfrac{n_{25}}{A}$ = fraction of 25-ft squares with values within A.

$f_{50} = \dfrac{n_{25}}{4n_{50}}$ = average fraction of 25-ft squares with values within each 50-ft square area, excluding blank 50-ft squares.

$f_{100} = \dfrac{n_{50}}{4n_{100}}$ = average fraction of 50-ft squares with values within each 100-ft square area, excluding blank 100-ft squares.

$f_{200} = \dfrac{n_{100}}{4n_{200}}$ = average fraction of 100-ft squares with values within each 200-ft square area, excluding blank 200-ft squares.

2.8.1 The nugget effects

The logarithmic error variances (nugget effects) for 50-ft and for larger squares can be estimated from (39):

$$N_{50} = \frac{1}{4f_{50}}\{N_{25} + (3a/2)\log\frac{1}{f_{50}}\}$$

$$N_{100} = \frac{1}{4f_{100}}\{N_{50} + (3a/2)\log\frac{1}{f_{100}}\} \tag{58a}$$

$$N_{200} = \frac{1}{4f_{200}}\{N_{100} + (3a/2)\log\frac{1}{f_{200}}\}.$$

In dealing with the straight averages of gold values within 50-ft, 100-ft and 200-ft squares it must, however, be borne in mind that for arithmetic sample means the logarithmic error variances are larger than those indicated by the usual formula of (population log variance)/n (see §1.10). Also, if the number of, for example, 25-ft squares within 50-ft squares averages, say, 3 ($f = 0,75$) but varies considerably from one 50-ft square to the next (i.e. from 1 to 4), this will also tend to increase the average logarithmic error variance for 50-ft squares calculated from the logarithmic error variance of the 25-ft squares. The following simple example will illustrate the combined effect of the two features mentioned:

Log error variance   = nugget effect for 25-ft squares
                      = $N_{25} = 0,4$.

Straight error variance = $m^2 (e^{0,4} - 1) = 0,4918$ m², absolute dispersion a = 0,02.

No. of 25-ft squares within 50-ft square (n)	1	2	3	4	Average 2,5 = $4f_{50}$
Frequency of 50-ft squares	25%	25%	25%	25%	—

Position on basis of untransformed values[†]:

Error variances of set of 25-ft squares	0,4918	0,2459	0,1639	0,1230	0,2562
Variance of this set within a 50-ft square*	0,0425	0,021	0,0087	—	—
Error variance of 50-ft square	0,5343	0,2669	0,1726	0,1230	0,2742

This average error variance of 0,2742 m² corresponds to an average logarithmic error variance N_{50} determined as follows:

$$0,2742 \text{ m}^2 = \text{m}^2 (e^{N_{50}} - 1),$$

$$N_{50} = 0,242,$$

whereas the straight application of (58a) would have given:

$$N_{50} = \frac{1}{4(0,625)} (0,4 + 0,03 \times 0,47)$$

$$= 0,1656.$$

To have given the value of 0,242, the f_{50} factor in (58a) would have had to be decreased from 0,625 to 0,439, that is, by some 30 %. Experience has shown that over the practical ranges of data patterns on the gold mines this adjustment can be as high as 40 %. Alternatively and preferably, the N_{50}, N_{100} and N_{200} values should be estimated directly from the variances for the 50-ft, 100-ft and 200-ft blocks respectively by application of (41) or (50), so that, for example

$$N_{25} = N_{50} + \sigma_{25}^2 - \sigma_{50}^2 - (3a/2)\log 4$$

$$N_{50} = \sigma_{50}^2 - \{\sigma_{25}^2 - N_{25} - (3a/2)\log 4\}, \tag{58b}$$

where $(3a/2)\log 4$ = variance of actual values of a 25-ft square within a 50-ft square,

$\sigma_{25}^2 - N_{25}$ = variance of actual values of 25-ft squares within the population, and hence

$\sigma_{25}^2 - N_{25} - (3a/2)\log 4$ = variance of actual values of 50-ft squares within the population (see (34)); and thus, from (58b),

N_{50} = (Observed variance of 50-ft square values) – (variance of actual values of 50-ft squares).

(see Fig. 23)

2.8.2 The population variances

From (40a):

$$\sigma_{25}^2 = N_{25} + (3a/2)\log A$$

$$\sigma_{50}^2 = N_{50} + (3a/2)\log (A/4)$$

$$\sigma_{100}^2 = N_{100} + (3a/2)\log (A/16) \tag{59}$$

$$\sigma_{200}^2 = N_{200} + (3a/2)\log (A/64).$$

The above relationships for σ_{25}^2 to σ_{200}^2 can be used where only one variance in the range σ_{25}^2 to σ_{200}^2 is known and where the straight line de Wijsian model provides a close fit over the whole range of areas and lags. *Where all these logarithmic variances can conveniently be observed in practice, and this is usually the case, the observed values should naturally be used directly.*

2.8.3 The covariances

$[\sigma_c^2]$ = logarithmic variance of actual values of central 100 ft blocks within population A,

$= \sigma_{100}^2 - N_{100}$,

= the logarithmic covariance between the estimates based on the 50-ft data blocks (A1 to A4 — Fig. 28) and the actual value of the central 100-ft block. (Matheron, 1960; Rendu, 1978)   (60)

[†]These variances all have to be multiplied by (population mean)².

*Note: From (39) and (58a) the increase in logarithmic variance is $(3a/2)\log (1/f_{50}) = (3a/2)\log a(4/n)$; the variance of the untransformed units will then be as above

Note that (60) is appropriate and confirms (18) provided the value of a 50-ft block can be accepted as a random sample drawn from, and as such is an unbiased estimate of, the population of values within the 100-ft ore block to be valued. This is a reasonable approximation where the individual sample values are spaced at random within the basic 25-ft squares and where anisotropy, if present (see §2.9), is weak or moderate. However, where the anisotropy is marked the covariances of the individual four central data blocks with the central 100-ft block will not be the same and should ideally be estimated directly from the semivariogram model by averaging the relevant covariances for smaller support areas, say, the constituent 25-ft squares. However, practical tests have shown that the direct use of the 50-ft semivariogram for this purpose provides an acceptable approximation even for very marked anisotropies.

The other covariances between the central block and the outside data blocks and between all the outside blocks are obtained from the semivariogram model for the 50 ft blocks and the observed population variance σ_{50}^2 using the relationship (24):

Covariance = population variance – semivariogram value. As above, practical tests have shown that even with a very marked anisotropy, no worthwhile improvements are obtained for these covariances even for short lags using the semivariogram based on the smaller constituent 25-ft areas and averaging these for the relevant blocks.

2.9 Anisotropic de Wijsian semivariogram and covariance patterns

In the South African gold reefs we usually find, as a sedimentological feature, ore 'shoots' in the general flow direction and trending in most areas in a northwesterly-southeasterly direction. Ore shoots would normally be irregularly-shaped, higher grade areas elongated in this generally preferred direction with lower grade areas in between. Both the 'shoots' and their direction(s) can range from well-defined to poorly-defined. In these circumstances the semivariogram patterns found along different directions usually disclose an *anisotropic pattern* defined by a fan of straight lines starting off at a common point (with the same nugget effect N) but with different slopes (or absolute dispersions α).

Along the preferred direction the slope of the semivariogram would be below the average trend — that of the *combined variogram* — indicating a slower fading of the covariance (or co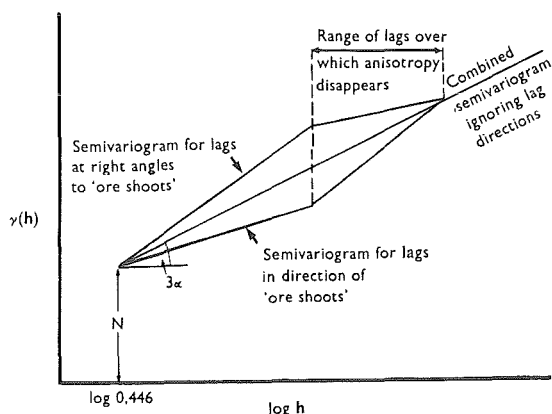rrelation) levels with increasing lag. At right angles to the preferred direction the slope would be steeper than average. In practice this anisotropic pattern could apply to a specific scale and over a limited range of lags only and would thus not necessarily be the same as the scale of analysis — thus, the size of the support (and possibly also of the field) is increased. In such an event it is obvious that any anisotropic pattern disclosed on a small scale will not necessarily extend over the full range of lags within the field; beyond a certain lag distance (which could be different for the various directions) the directional variogram could again gradually merge with the combined variogram.

These features are shown in Fig. 29. Where such changes in anisotropy are significant and occur within lag distances which will be used for kriging estimates, allowance should be made for these changes in the covariance matrix used.

If the differences in the x and y coordinates of the centres of two square support areas are accepted as, say, x and y, then

$$\log \text{ of lag } \mathbf{h} = \log \sqrt{(x^2 + y^2)},$$

but as the fan of semivariogram values start from a common point where $\gamma(\mathbf{h})$ = nugget effect, and as this occurs at

$$\log \mathbf{h} = \log 0{,}446, \qquad \text{see (45)}$$

the lag unit to be used for model fitting will be

$$\mathbf{h}_i = \log \frac{\sqrt{(x^2 + y^2)}}{0{,}446}. \qquad (61)$$

The direction ($\tan \varDelta$) for this lag will be (y/x).

2.9.1 *Formula for anisotropic semivariogram*

Corresponding to the combined semivariogram for square support areas,

$$\gamma(\mathbf{h}) \simeq N + 3\alpha \log \frac{\mathbf{h}}{0{,}446}, \qquad \text{see (44)}$$

the anisotropic semivariogram is defined by:

$$\gamma(\mathbf{h}_i) \simeq N + 3\alpha f \mathbf{h}_i, \qquad (62)$$

where N = nugget effect   ⎫ for the 'combined' semi-
⎬ variogram ignoring lag
α = absolute dispersion ⎭ directions

and f = a factor which, for any specified value of $\mathbf{h}_i$, will average unity over all directions but will be less than unity (and reach a minimum) for lags in the preferred ore 'shoot' direction, and more than unity (and reach a maximum) for lags at right angles to this direction.

2.9.2 *Estimation of N, the nugget effect, and 3α*

N corresponds to the $\gamma(\mathbf{h})$ value at the common intersection point of all the straight semivariogram lines fitted by least squares to the observations $\gamma(\mathbf{h}_{ij})$ in all the directions j observed and corresponding to the $\mathbf{h}_i$ values concerned for each j direction,

$$\text{where} \qquad \mathbf{h}_{ij} = \log \frac{\sqrt{(x^2 + y^2)}}{0{,}446}$$

and $\gamma(\mathbf{h}_{ij}) = \gamma_{ij} \simeq N + 3\alpha (\mathbf{h}_{ij}).$

The least squares solution for N is given by

$$N = \frac{\overset{j}{\underset{}{\varSigma}} \overset{i}{\underset{}{\varSigma}} w_{ij} \gamma_{ij} - \dfrac{(\overset{i}{\varSigma} w_{ij} \mathbf{h}_{ij})(\overset{i}{\varSigma} w_{ij} \gamma_{ij} \mathbf{h}_{ij})}{\overset{i}{\varSigma} w_{ij} \mathbf{h}_{ij}^2}}{\overset{i}{\underset{}{\varSigma}} \overset{j}{\underset{}{\varSigma}} w_{ij} - \overset{j}{\underset{}{\varSigma}} \dfrac{(\overset{i}{\varSigma} w_{ij} \mathbf{h}_{ij})^2}{\overset{i}{\varSigma} w_{ij} \mathbf{h}_{ij}^2}}. \qquad (63)$$

Fig. 29. The logarithmic-de Wijsian semivariogram with anisotropy not maintained.

This solution is identical with fitting directly by least squares a straight line variogram to all the observed values $\gamma(\mathbf{h}_i)$ of the combined semivariogram as ordinates with the corresponding $\mathbf{h}_i$ values as the abscissae and solving for the value of the line function for $\mathbf{h}_i = 0$. Equation (63) then reduces to:

$$N = \frac{\Sigma w_i\, \mathbf{h}_i\, \Sigma w_i\, \mathbf{h}_i\, \gamma(\mathbf{h}_i) - \Sigma w_i\, \gamma(\mathbf{h}_i)\, \Sigma w_i\, \mathbf{h}_i^2}{\left(\Sigma w_i\, \mathbf{h}_i\right)^2 - \Sigma w_i\, \Sigma w_i\, \mathbf{h}_i^2}, \qquad (64)$$

which is the standard least squares solution for the y-intercept c of the straight line equation $y = mx + c$. The slope m of this straight line corresponds to 3α in the standard formula for the logarithmic–de Wijsian semivariogram (combined — ignoring lag directions) and is estimated by the formula:

$$3\alpha = \frac{\Sigma w_i\, \Sigma w_i\, \mathbf{h}_i\, \gamma(\mathbf{h}_i) - \Sigma w_i\, \mathbf{h}_i\, \Sigma w_i\, \gamma(\mathbf{h}_i)}{\Sigma w_i\, \Sigma w_i\, \mathbf{h}_i^2 - \left(\Sigma w_i\, \mathbf{h}_i\right)^2}. \qquad (65)$$

Theoretically, the weights w_i or w_{ij} above will be the observed frequencies of the pairs of values used in calculating $\gamma(\mathbf{h}_i)$ or γ_{ij}. In practice, as the length of lag increases, the frequencies of pairs also increase to very large numbers compared to the shorter lags (see Fig. 24). As the shorter lags are more important in kriging operations a least squares fit based on frequencies over the full range of lags could in practice overemphasize the longer lags and result in unrealistic estimates for N and 3α. Furthermore, the physical data distribution pattern could be such as to inflate the number of pairs in a particular direction j or certain consecutive j directions, and as this direction(s) could be linked positively or negatively with any preferred 'ore shoot' direction, this could introduce further biases in estimating N and 3α from (64) and (65), and even in the population variance. Hence, in determining N and 3α *weighting is not recommended* over the segments or the lag categories, provided the latter are spread along the logarithmic scale on a reasonably even basis (see §3.37 for a case study), and provided the numbers of pairs of values all exceed, say, 600. Where the data are limited, weighting should be used subject to a maximum weight corresponding to, say, 200 pairs of values.

The straight line is thus fitted directly to the unweighted combined semivariogram value for each of the lag categories up to the maximum lag to be used in the kriging procedures. Furthermore, a graphical plot is essential and, if towards the longer lag categories a significant departure from the straight line model is evident, the model should be fitted exclusive of the longer lags, and an appropriate adjustment to the model can then be incorporated for such lags (see §3.3 for a detailed case study).

2.9.3 Form of the anisotropic function f

Experience indicates that f can be accepted as the 'radius' of an ellipse with semi-axes a and b. Then

$$f = \left\{ \frac{a^2\, b^2}{a^2 \sin^2\beta + b^2 \cos^2\beta} \right\}^{\frac{1}{2}} \qquad (66)$$

with $\bar{f} = 1$, for β values over the full range from $0°$ to $180°$, and $\beta =$ angle between the radius and the major semi-axis a. The semivariogram function (62) can then be defined by $\gamma(\mathbf{h})$:

$$\gamma(\mathbf{h}) \simeq N + 3\alpha \left\{ \frac{a^2\, b^2}{b^2 \cos^2(\varDelta - \theta) + a^2 \sin^2(\varDelta - \theta)} \right\}^{\frac{1}{2}} \log \frac{\mathbf{h}}{0,446} \qquad (67)$$

where $\varDelta =$ angular direction of lag $\mathbf{h}$ and $\tan \varDelta = (y/x)$,

$\theta =$ angular direction of major axis of ellipse, i.e. direction at right angles to 'ore shoot' direction, and

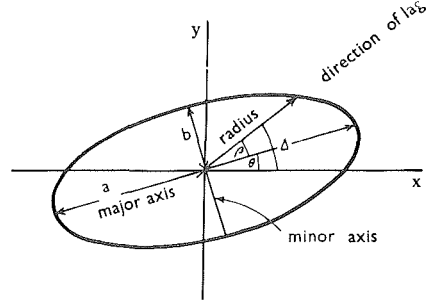$\varDelta - \theta = \beta =$ angle between lag direction and major axis of ellipse.

Fig. 30. The model used for elliptical anisotropy.

For isotropy, $a = b = 1$ and $f = 1$.
For lags in the direction of the major axis, that is, at right angles to ore shoots:
$\theta = \varDelta$, $\varDelta - \theta = 0$, and $f = a$, its maximum value.
For lags in direction of minor axis, that is, in direction of ore shoots:
$\theta = \varDelta + 90°$, $\varDelta - \theta = -90°$, and $f = b$, its minimum value.

These features are shown graphically in Figs 30 and 31.

Note that this model for anisotropy is based on an elliptical pattern for the semivariogram values corresponding to specific lags, whereas that suggested by David (1977) and others is based on an elliptical pattern for the lags corresponding to specific semivariogram values. The latter approach is open to criticism as the combined semivariogram represents in effect the regression of semivariogram values γ against lag $\mathbf{h}$ and not the complementary regression, which is of $\mathbf{h}$ on γ. In practice, results from these two approaches are likely to be similar where the anisotropy remains stable. However, where it changes at certain lag distances rather than at certain semivariogram values, the approach described here seems preferable.

2.9.4 Estimation of parameters a, b and θ

A simple graphical basis for estimation is to calculate the average semivariogram values and the corresponding average log $(\mathbf{h}/0,446)$ values for specified directions (say 8 represented by 8 segments of a semi-circle) over the full range of lags, or preferably up to some specified lag (see note at end of §2.9.2 above).
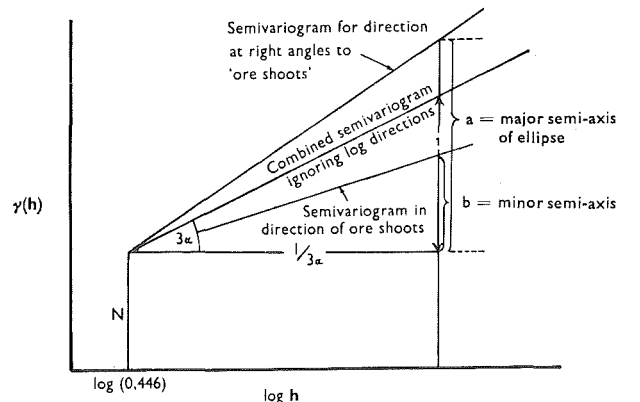
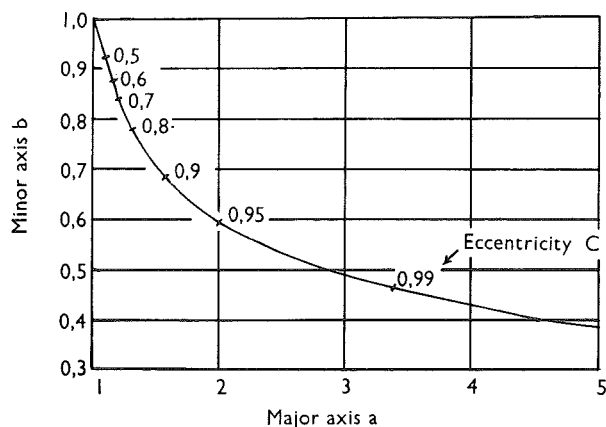Fig. 31. Anisotropic logarithmic-de Wijsian semivariogram for gold values.

Fig. 32. The relationships between the axes of an ellipse and its eccentricity corresponding to a unit mean radius for the ellipse.

From (67):

$$f = \frac{\gamma(\mathbf{h}) - N}{3\alpha \log(\mathbf{h}/0,446)}.$$ (68)

From the average $\gamma(\mathbf{h})$ and $\log(\mathbf{h}/0,446)$ values for each direction over the selected range of lags and the previously determined values for N and α, an estimated value for f can be obtained for each direction and plotted graphically against the corresponding angle of direction Δ (on the x-axis). To these points a smooth curve can then be fitted representative of the function f as defined in (66) and (67) above.

For f to average unity the parameters a and b and the eccentricity c of the ellipse are interrelated as shown in Fig. 32. Secondly, the shape of function f over the range of values for the angles $\beta = (\Delta - \theta)$ from $0°$ to $180°$ will be fixed for any specified eccentricity c (and hence for every specified pair of values for a and b) as shown in Fig. 33. The set of curves from this diagram is, therefore, superimposed on the plotted f values and shifted horizontally to provide a visual 'best' fit of one of the curves to the plotted f values. This fit will provide the required estimates for:
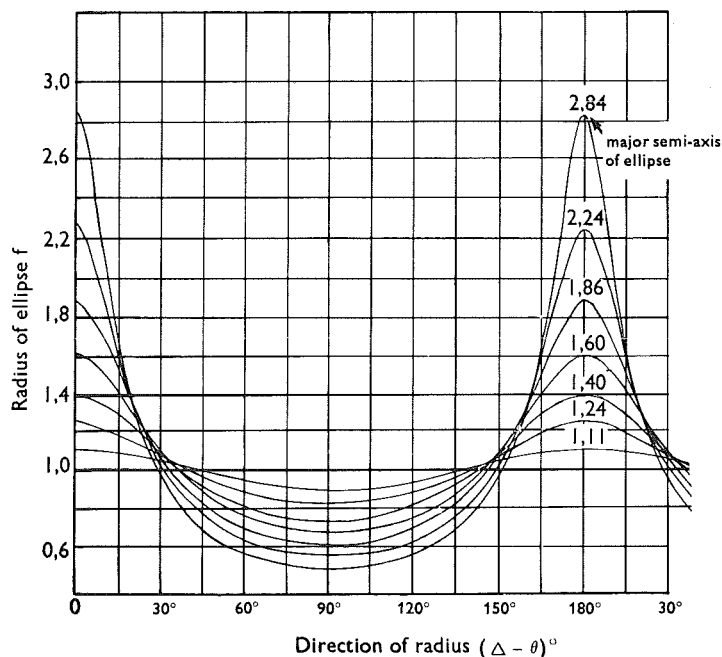
(i) the parameters a and b and the eccentricity c, and

Fig. 33. The relationship between the radius f of an ellipse and its direction for a range of major axes.

(ii) the angle θ, being the position on the horizontal Δ scale coinciding with the peak of the fitted f curve (when angle $\beta = 0$). The coordinate system used measures angles anticlockwise with zero west.

The author prefers this graphical fit of the model to any more sophisticated mathematical fit, because the appropriateness or otherwise of the model in representing the data will usually be evident as well as the reliability of the estimates of the relevant parameters.

The attention given in this section to anisotropy is justified by the fact that, where it is significant, *its recognition, proper definition and introduction into ore valuation procedures will always yield significant improvements in efficiency* (see §2.12.1).

2.10 Kriging solutions for normally distributed or normalised values based on normal multiple regression theory

(Matheron, 1960; Rendu, 1978)

2.10.1 *Population mean known*

This is the standard linear multiple regression solution for the dependent normal variable y given n independent normal but correlated variables x_i.

Let y = actual value of ore unit to be estimated,

x_i (or x_j) = values of n data ore units available,

a_i (or a_j) = optimum weights for the x_i (or x_j) data values,

σ_{ij} = covariance between x_i and x_j data values,

σ_{yi} = covariance between y and x_i (or x_j) values,

and M = population mean.

Then the optimum linear estimator will be

$$a_1 x_1 + a_2 x_2 + \ldots a_i x_i + \ldots a_n x_n + (1 - \Sigma a_i)M$$ (69)

and its error variance

$$= \sigma_e^2 = \text{kriging variance } (\sigma_k^2)$$

$$= [\sigma_y^2] - \overset{i}{\Sigma} a_i \sigma_{yi}.$$ (70)

The solution for the optimum weights a_i and hence for the error variance is given in matrix notation by:

$$A = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ & & & \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ & & & \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{bmatrix} \quad X = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad B = \begin{bmatrix} \sigma_{y1} \\ \sigma_{y2} \\ \vdots \\ \sigma_{yn} \end{bmatrix}$$ (71)

with $X = A^{-1}B$,

and where the σ_i^2, σ_{yi} and σ_{ij} values are in practice replaced by their estimators S_i^2, S_{yi} and S_{ij} respectively. Note that in the data matrix A to be inverted, the S_i^2 *values are the observed variances — they are inclusive of the nugget effect.*

2.10.2 *Population mean unknown or ignored*

The estimator then changes to

$$a_1 x_1 + a_2 x_2 + \ldots + a_i x_i + \ldots \ldots \ldots a_n x_n$$
$$\text{with } \Sigma a_i = 1.$$

The error variance $= \sigma_e^2 = \sigma_k^2 = [\sigma_y^2] - a_i \sigma_{yi} + \mu,$ (72)

where μ = La Grange multiplier (positive) and the solutions for μ and the optimum weights a_i are given by:

$$
A = \begin{bmatrix}
\sigma_1^2 & \sigma_{12} & \cdots\cdots & \sigma_{1n} & -1 \\
\sigma_{21} & \sigma_2^2 & \cdots\cdots & \sigma_{2n} & -1 \\
\cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & & \cdot & \cdot \\
\sigma_{n1} & \sigma_{n2} & \cdots\cdots & \sigma_n^2 & -1 \\
1 & 1 & \cdots\cdots & 1 & 0
\end{bmatrix}
\quad
X = \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_n \\ \mu \end{bmatrix}
\quad
B = \begin{bmatrix} \sigma_{y1} \\ \sigma_{y2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \sigma_{yn} \\ 1 \end{bmatrix}
\quad (73)
$$

with $X = A^{-1}B$,

and the σ^2 and σ values are replaced by their estimators as above.

2.11 Lognormal kriging solutions

(References: Matheron, 1960; Rendu, 1978; Marechal, 1974)

Where the ore values are lognormally distributed and the proportional effect is therefore present, the ideal approach is to transform and normalise the ore values by taking logs (after adding a constant if necessary), establish the variogram pattern for these transformed values, arrive at the corresponding kriging solution for the weights and the estimator, and finally retransform this kriged estimator back into the untransformed unit (with deduction of the additive constant where applicable). The y^*, z^*, x_i, and M values below will therefore all include the additive constant if applicable.
Let:

$\log y^*$ = kriged estimator of $\log y$ based on the transformed values,

z^* = kriged estimator of the untransformed true grade z of the ore block to be valued,

x_i = grades of data ore blocks,

a_i = kriging weights for the transformed values for x_i,

σ_{yi} = covariance between transformed values of y and x_i, and

M = population mean of x_i values.

2.11.1 Population mean known

The estimator on the transformed basis, $\log y^*$, corresponding to §2.10, is given by:

$$(\log y^* - \overline{\log y}) = \Sigma a_i (\log x_i - \overline{\log x_i}) \quad (74)$$

but for the lognormal case (see §1.8):

$$\log \bar{y} = \overline{\log y} + \tfrac{1}{2} [\sigma_y^2]$$

$$\log \bar{x}_i = \overline{\log x_i} + \tfrac{1}{2} \sigma_{x_i}^2$$

and since $\quad \bar{y} = \bar{\bar{x}}_i = M \qquad$ as in §2.10.1,

$$\overline{\log y} = \log M - \tfrac{1}{2} [\sigma_y^2]$$

$$\overline{\log x_i} = \log M - \tfrac{1}{2} \sigma_{x_i}^2$$

$$(75)$$

(74) then reduces to

$$\log y^* = (1 - \Sigma a_i) \log M + \Sigma a_i \log x_i$$
$$- \tfrac{1}{2} \{[\sigma_y^2] - \Sigma a_i \sigma_{x_i}^2\} \quad (76)$$

and since the error variance for $\log y^*$

$$= \sigma_{ye}^2 = [\sigma_y^2] - \Sigma a_i \sigma_{yi} \quad \text{(see (70))} \quad (77)$$

and the estimator z^* on the untransformed basis is related to y^* (see §1.17) by:

$$\log z^* = \log y^* + \tfrac{1}{2} \sigma_{ye}^2, \quad (78)$$

$$\log z^* = (1 - \Sigma a_i) \log M + \Sigma a_i \log x_i$$
$$+ \tfrac{1}{2} \Sigma a_i (\sigma_{x_i}^2 - \sigma_{yi}). \quad (79)$$

With σ_{ye}^2 replaced by its estimate S_{ye}^2, the error variance for $\log z^*$ will be slightly larger than that for $\log y^*$ in (77). Unless the data used are very limited this effect will usually be negligible.

2.11.2 Solution for mean unknown

In this case the error variance for $\log y^*$

$$= \sigma_{ye}^2 = [\sigma_y^2] - \Sigma a_i \sigma_{yi} + \mu \quad \text{(see (72))} \quad (80)$$

and the estimator z^* is then defined by:

$$\log z^* = \Sigma a_i \log x_i + \tfrac{1}{2} \Sigma a_i (\sigma_{x_i}^2 - \sigma_{yi}) + \tfrac{1}{2}\mu. \quad (81)$$

Again the solution for the error variance of $\log z^*$ has not been attempted but will be somewhat larger than that indicated by (80).

2.11.3 Suboptimum weights

Where suboptimum weights are used the error variance from orthodox statistics will be (see also David, 1977, §4.4.1):

$$\sigma_{ye}^2 = [\sigma_y^2] + \underset{i\ j}{\Sigma \Sigma} a_i a_j \sigma_{ij} - 2 \Sigma a_i \sigma_{yi}. \quad (82)$$

In line with (76), (78), (79) and (81), the estimator in this case (z^φ) will be defined by:
(i) Mean known:

$$\log z^\varphi = (1 - \Sigma a_i) \log M + \Sigma a_i \log x_i$$
$$+ \tfrac{1}{2} \Sigma a_i (\sigma_{x_i}^2 - 2 \sigma_{yi}) + \tfrac{1}{2} \underset{i\ j}{\Sigma \Sigma} a_i a_j \sigma_{ij}. \quad (83)$$

(ii) Mean unknown (with $\Sigma a_i = 1$):

$$\log z^\varphi = \Sigma a_i \log x_i + \tfrac{1}{2} \Sigma a_i (\sigma_{x_i}^2 - 2 \sigma_{yi})$$
$$+ \tfrac{1}{2} \underset{i\ j}{\Sigma \Sigma} a_i a_j \sigma_{ij}. \quad (84)$$

It must be stressed again that the limits of error corresponding to all the above logarithmic error variances will be skew (see §1.10 and Table 4) and can be accepted as symmetrical only if the error variance is very small ($\ll 0{,}005$).

2.12 Practical considerations in applying kriging procedures

The application of simple kriging procedures for ore valuation requires decisions on
(i) how and down to what scale to subdivide the ore body in order to ensure reasonable homogeneity and stationarity within each subdivision;
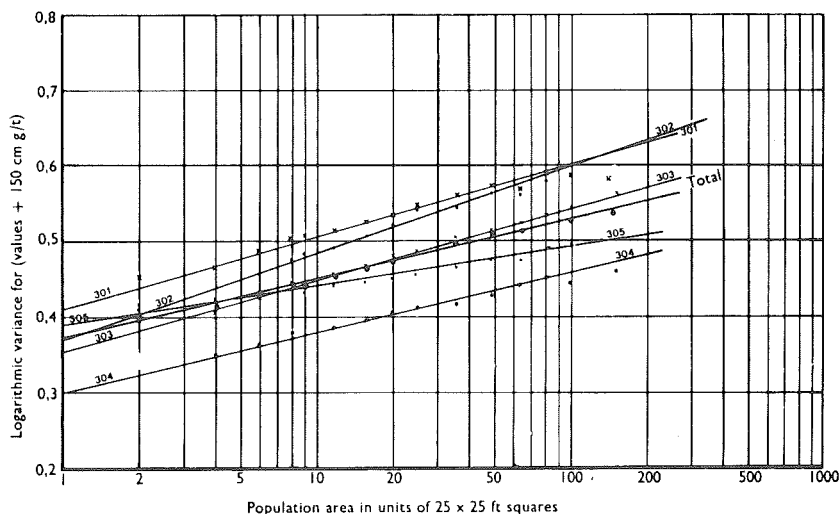
Fig. 34. The variance-size of area relationships for subsections within a section of the Hartebeestfontein mine.

(ii) the maximum lag up to which data should be used for kriging and whether or not the mean grade of the mine section should be brought into the calculations or ignored; and

(iii) the minimum size of ore block which should be valued by these procedures.

2.12.1 *Size and boundaries of mine sections*

A recent investigation was carried out on the gold values for a large section of the Hartebeestfontein mine in the Klerksdorp goldfield (Krige, 1976). This analysis was aimed at establishing the nature and extent of variations in the de Wijsian semivariogram and other relevant parameters for smaller parts of this section, and, with this information, to determine the extent of any losses in efficiency by accepting homogeneity for the section as a whole instead of subdividing it into smaller

and individually more homogeneous subsections first.

The variations in the variance-size of area and the semivariogram patterns for 5 subsections numbered 301-5 (540 000 m² each) and for the whole section (2 700 000 m²) are shown in Figs 34 and 35 respectively. The semivariograms based on 25-ft squares were all accepted as reasonably isotropic except for subsection 302 where a significant anisotropy (approximately N-S ore shoot trend) was evident. Accepting a specified pattern of known data for each 100-ft square ore block to be valued, the error (or kriging) variance for grade estimates of such blocks was calculated on the semivariogram model for the mine section as a whole, as well as on the model for the relevant subsection. Two sets of kriging weights were thus calculated for each subsection based on the parameters for (i) the whole mine section, and (ii) for the relevant subsection respectively. For these the
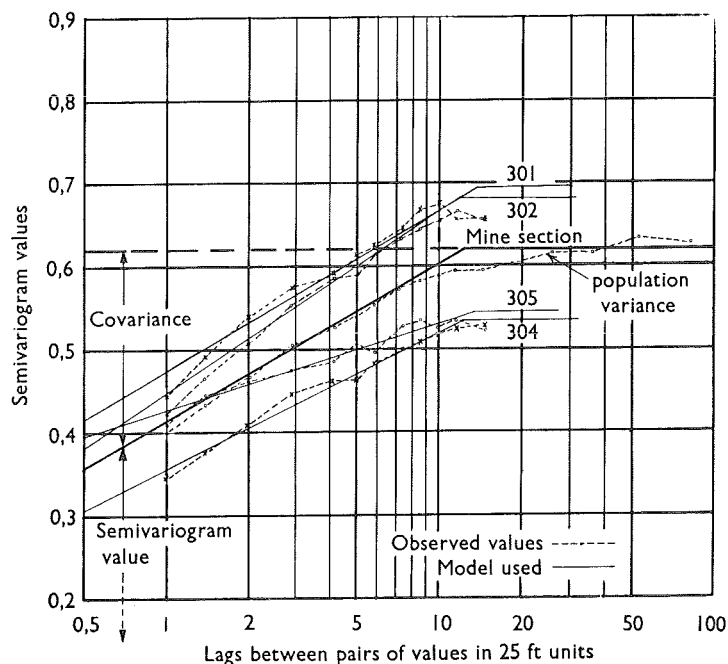

Fig. 35. Variograms for a section and for subsections 301, 302, 304 and 305 of the Hartebeestfontein mine.

Table 5

Effect on weighting systems and error variances of different parameters used

| | Mine section | Subsections on isotropic basis | | | | | 302 Accepting anisotropy | |
		301	302	303	304	305	Data E-W	Data N-S
Parameters used (25-ft squares)								
Population variance	0,620	0,695	0,680	0,620	0,535	0,545	0,680	0,680
Nugget effect	0,372	0,410	0,370	0,355	0,300	0,390	—	—
Absolute dispersion	0,023	0,028	0,033	0,027	0,023	0,015	0,033	0,033
Optimum weights								
Data block 1	0,118	0,117	0,112	0,115	0,119	0,103	0,066	0,103
2	0,169	0,170	0,174	0,171	0,174	0,146	0,257	0,084
3	0,169	0,170	0,174	0,171	0,174	0,146	0,257	0,084
4	0,118	0,117	0,112	0,115	0,119	0,103	0,066	0,103
5	0,015	−0,018	−0,002	−0,018	−0,015	−0,006	0,001	−0,015
6	0,023	0,015	−0,020	0,002	0,014	0,022	−0,044	−0,044
7	0,015	0,005	−0,013	−0,003	0,002	0,026	0,031	−0,023
8	0,015	0,005	−0,013	−0,003	0,002	0,026	0,031	−0,023
9	0,023	0,015	−0,020	0,002	0,014	0,022	−0,044	−0,044
10	0,015	−0,018	−0,002	−0,018	−0,015	−0,006	0,001	−0,015
Population mean	0,321	0,423	0,498	0,466	0,411	0,416	0,380	0,792
Error variance of estimates (log basis)								
On subsection weights	0,136	0,162	0,184	0,154	0,132	0,090	0,156	0,215
On mine section weights	0,136	0,164	0,187	0,157	0,134	0,092	0,165	0,241

corresponding error variances were determined accepting the semivariogram models for the subsections as correct — that is, accepting (ii) as the optimum kriging weights and (i) as suboptimum weights.

The results are tabulated in Table 5 and show that the kriging procedures are very robust and that in these circumstances the use of suboptimum weights based on the mine section as a whole results in an insignificant loss of efficiency; also that *subdivision into smaller mine sections would be warranted only where there are significant changes in anisotropy* from subsection to subsection. This conclusion will not necessarily apply in all cases and for meaningful decisions in practice, therefore, an analysis of this type is recommended for every ore body.

Information on geological features such as major faults, major changes in ore width and observed patterns for various sedimentological features, can assist materially and should naturally also be taken into consideration in decisions on section sizes and boundaries (Krige and others, 1969) in order to ensure reasonable homogeneity within mine sections.

2.12.2 *Size of data area*

It is obvious that if the larger area covering the surrounding data used for kriging estimates of an ore block within such an area cannot itself be valued on the data within close limits of error, it will itself be subject to error and bias due to regression effects (see §1.17–1.21). *In such a case efficiency will be lost and conditional biases will not be eliminated entirely, unless the known mean value for the mine section is also introduced into the kriging solution.*

A general principle which should, therefore, be applied is that *kriging without the population mean* should extend to and include all data within a minimum size of area which can itself be valued without significant error and bias on this data (or to the whole field or population area where the grade estimate for the field is still subject to significant error). Alternatively, *kriging with the population mean* should relate

to 'fields' or population areas of this same minimum size but data used within these fields could then be limited to, say, two 'peripheries'. As shown by Matheron (1960) and confirmed by the author (Krige, 1966a) in practical analyses, two peripheries normally provide an effective 'screening' of additional data outside such peripheries (see also David, 1977). The practical results from the general approach outlined above should agree closely with those obtained from universal kriging (Matheron, 1971).

This problem can be compared with that of erecting a tall building on ground which is unstable and has a high relief — which is equivalent to valuing an ore block on the evidence of widely spaced and variable ore grades. Stability (good estimates) can be provided by erecting the building on a solid flat base extending to an area (the data field or population area) just large enough to provide the required firmness of support (the lowest kriging variance for the estimate). Any additional enlargement of such an area would not significantly improve the firmness of support and could actually raise or lower the whole base (introduce local biases where the population mean used relates to an unnecessarily large area or field — Krige, 1963). In universal kriging, which is not discussed here, the equivalent of the broad support would be provided by a firm undulating surface sheet overlying the unstable ground and satisfactorily reflecting the major relief features of the ground.

2.12.3 *Size of ore block to be valued*

The minimum size of block to be valued by these procedures should in practice not be smaller than about *a quarter of the average interval of the data grid* (see David, 1977, §10.2.1). Valuation of smaller blocks will substantially increase the effort involved and will not provide any significant increase in the detail of the valuation pattern as adjacent smaller blocks will tend to have very similar grade estimates. When the data are highly variable, such as in gold mines, the minimum size will be substantially larger.

Geostatistical Models and Case Studies
for Certain Gold Mines and the Prieska Copper Mine

3.1 Computer programs

For practical reasons all basic input data for the gold mines of the Anglovaal Group conform to a regular grid of averages of sample values within 25-ft squares in the plane of the ore body; individual sample values with their metric co-ordinates are input and converted by a computer program to the basic 25-ft square averages. At the Prieska copper mine the borehole data are forced on to a regular grid of 15-m squares corresponding to the coordinates of the grid points nearest to the actual borehole intersections. As for the gold mines the Prieska ore body, although some 3 to 30 m thick, is also dealt with on a two-dimensional basis (Krige, 1973).

On the gold mines the data subjected to geostatistical analysis cover: gold values (in cm g/t), uranium values (cm kg/t), and channel widths (cm). The gold and uranium values are normalised on the basis of log (value plus constant) and all the geostatistical analyses are performed on these transformed values. The channel widths usually conform reasonably well to the normal distribution and are, therefore, not transformed.

At Prieska the metal grades analysed cover the two main products, copper and zinc, as well as the impurities lead and bismuth; in addition the specific gravity values and the hanging- and foot-wall deviations from a projection plane are also handled. These were all previously used on an untransformed basis but because of problems associated with the proportional effect the lognormal-de Wijsian model is now used for some of the grade variables. The key computer programs can be summarised briefly as follows:

3.1.1 Gold mines

(i) Calculation of 50, 100 and 200-ft square averages from basic 25-ft squares values; frequency distributions (for determination of the additive constant); and the variances of the normalised values.

(ii) Average logarithmic variances within a range of subdivisions of each mine section to establish the variance-size of area relationships.

(iii) Semivariogram calculations for a range of lag categories and for 4, 8 or 16 segments of a semicircle to cover 4, 8 or 16 directions (usually 8); where the data are limited all possible pairs are used, otherwise data can be selected for analysis on a specified grid (close for short lags and wider for longer lags).

(iv) Fitting of the combined semivariogram, pooled over all directions, and up to a specified lag; also the calculation of the f values for the specified directions to enable the graphical determination of the parameters for the anisotropic semivariogram.

(v) Setting up the complete covariance matrices corresponding to specified data patterns, the inversion of these to enable the regression coefficients (kriging weights) to be established for each data pattern, and the determination of standard weighting tables (Krige, 1966).

(vi) Calculation of the weighted moving averages (kriged estimates) of standard 100 ft × 100 ft areas on a 100-ft grid.

(vii) Contour plots of the kriged gold, uranium and width values to enable meaningful ore blocks to be manually delineated.

(viii) Digitizing of the ore block coordinates.

(ix) Integration of the kriged gold, uranium and width estimates for the standard 100 ft × 100 ft areas or portions of these falling within each ore block, and calculation of ore tonnages, average grades, widths and metal contents. For the theoretical justification of pooling kriged estimates over larger ore blocks, see David (1977, p. 257).

(x) Ore reserve tabulations of tons, grades etc. for all ore blocks in mine sections (individual and combinations), grade categories etc.

3.1.2 Prieska (Krige, 1973, 1975)

(i) Data base for survey, geological, structural, mineralogical and analytical data for all boreholes and other underground intersections.

(ii) Selection of the foot- and hanging-wall positions for each borehole for optimum profit at specified metal prices and costs; calculation of the metal grades and specific gravity over this selected width and of the three-dimensional coordinates of the foot- and hanging-wall intersections.

(iii) Fitting by a modified form of universal kriging surfaces to all the foot- and hanging-wall intersections to enable widths and tonnages to be calculated and detailed sections to be drawn for mine planning (Krige, 1975).

(iv) Gridding of the metal grades and specific gravity data from (ii), and detailed statistical analyses (tests for outliers, calculations of means, variances, histograms, cross-correlations, etc.) and geostatistical analyses (semivariograms, variance-size of area relationships etc.) of the data on the basis of either untransformed or transformed values, whichever is appropriate.

(v) The fitting of a selected semivariogram model (spherical, exponential, or de Wijsian etc.) to the data (untransformed or transformed).

(vi) Kriged estimates based on the model selected under (v).

(vii) Contour plots of grades, specific gravity, widths, hanging- and foot-wall surfaces, and combined profit per ton or per unit area.
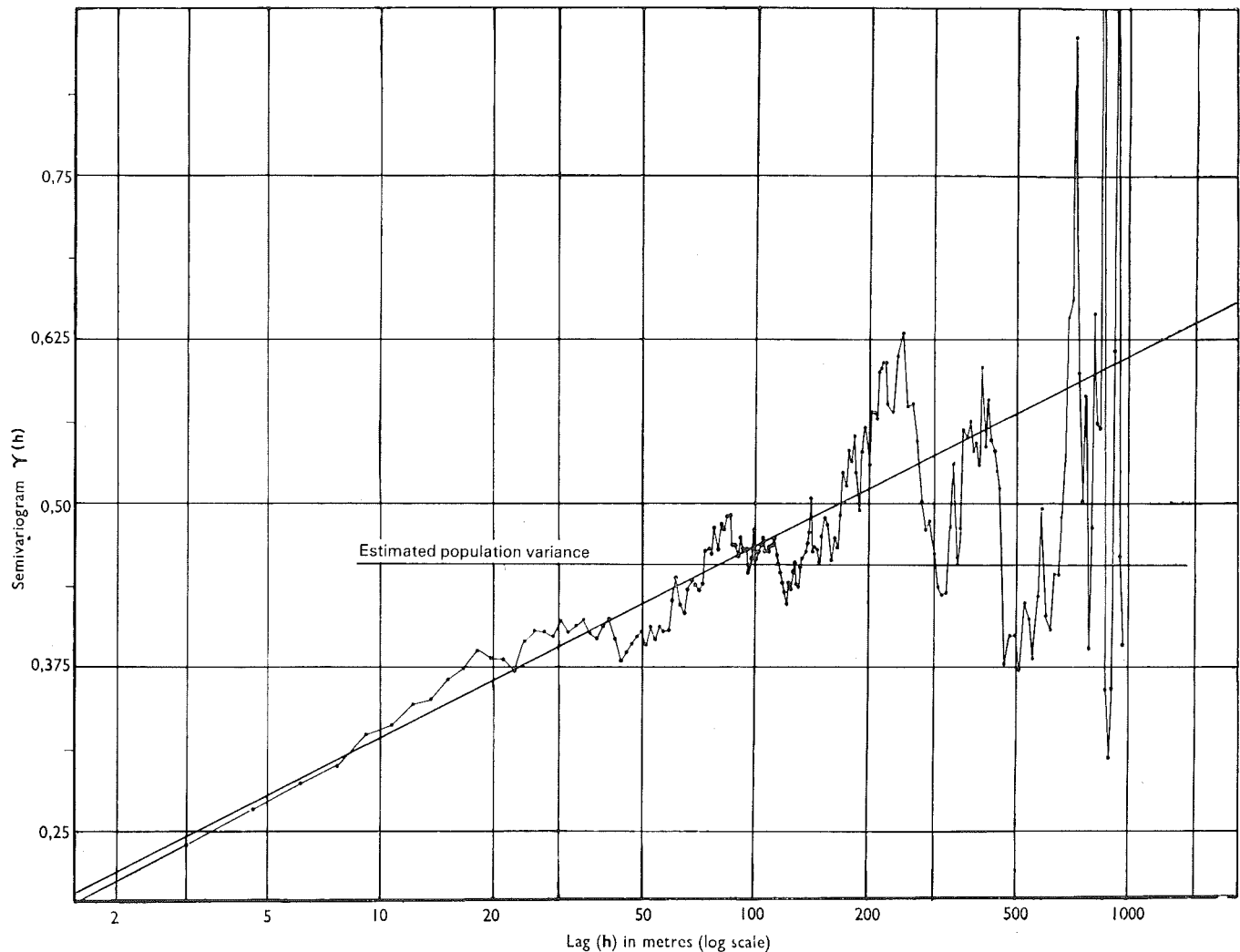
Fig. 36.   The observed semivariogram pattern for gold values on the Leader Reef in the President Steyn gold mine.

(viii) Integration of these estimates over individual manually delineated ore blocks and the tabulation of the ore reserves in various categories.

3.2 De Wijsian semivariogram for gold values over a large field

A geostatistical analysis for an entire gold mine or for a major part of one is not common, because the property is usually subdivided into smaller, more homogeneous sections for practical and theoretical reasons. Recently, however, an analysis was carried out of all the gold cm g/t values available on the Leader Reef on the President Steyn gold mine in the Orange Free State. This reef was not being mined but had been explored by a limited amount of development spread over a large part of the mine on a very irregular pattern. The development involved mainly driving augmented with some raising, and the pattern of data distribution was such that a proper test for anisotropy of the semivariogram could not be effected. However, the logarithmic semivariogram for all the N-S drive values (Fig. 36) shows a reasonable overall agreement with the de Wijsian model over the full range of lags from 3 to almost 1000 metres; the area covered was some 5 million m² and the values analysed totalled some 2000.

It is clear from the diagram that although the semivario-

gram would appear to level off or even reverse its trend at certain lag intervals, it again returns to the overall linear trend for longer lag intervals. When plotted on an arithmetic lag scale the cyclic deviations from the overall trend provide a good example of the so called *hole-effect*. In this case it is probably due to the presence over this large area of richer and poorer zones or areas repeated in fairly regular cycles at intervals of about 50, 125, 300 and 500 metres and resulting in deviations from the de Wijsian model in the vicinities of these lags.

It is worth noting that, when plotted on an arithmetic lag scale, the data can be well represented by the *spherical semivariogram* model (Rendu, 1978; David, 1977) up to lags of about 50 metres *but with a sill well below the population variance*: however, for greater lags this model will be quite unsuitable.

3.3 Case studies for sections of the Hartebeestfontein gold mine in the Klerksdorp goldfield, and the Loraine mine in the Orange Free State

In these studies the notations used in Section 2 for population parameters, for example σ^2 for variance, have, for the sake of simplicity, been applied to the estimates of these parameters based on the available data. The relevant section of the
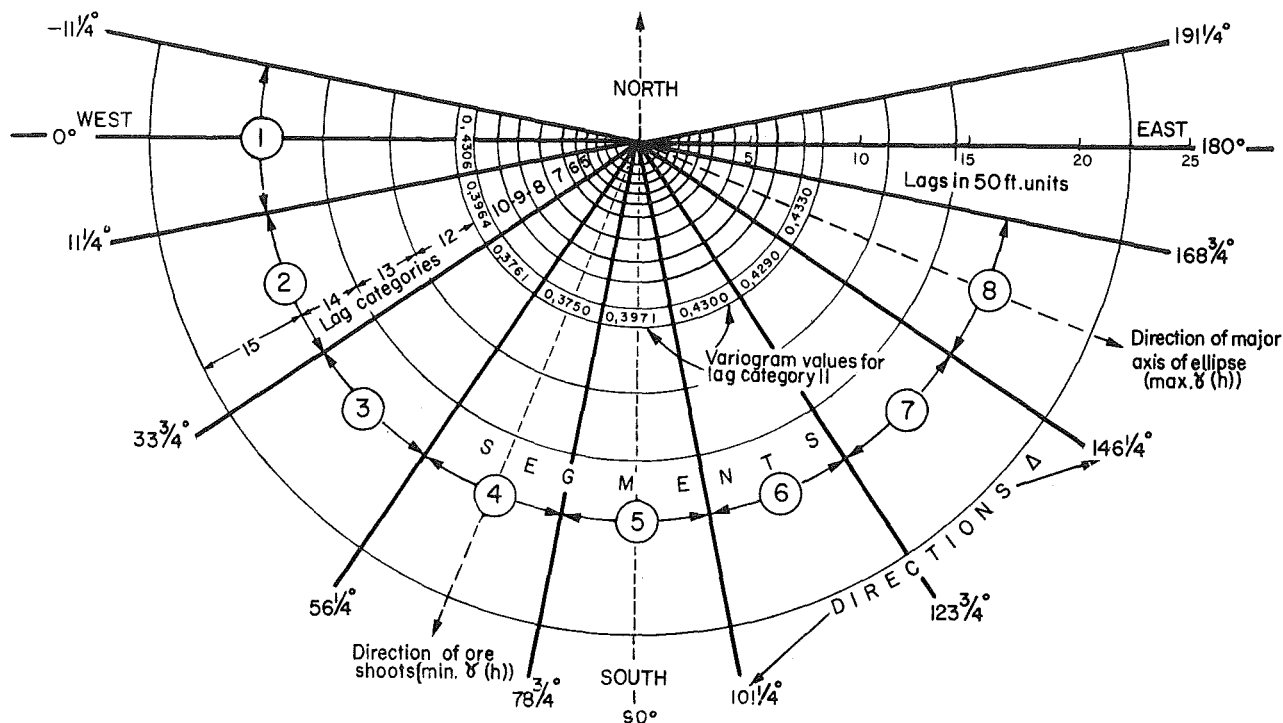
Fig. 37.   Segments and lag categories for the variogram analysis of a section of the Hartebeestfontein mine.

Hartebeestfontein mine covers an area of some $12\frac{1}{2}$ million sq ft (1,2 million m²) and had data available in stoped-out areas and in development from a total of 11 752 data squares (25 ft × 25 ft). The analyses performed are discussed in §3.3.1–3.3.5.

3.3.1 Variances

The cm g/t gold values were normalised on the basis of log (value + 150) and the corresponding logarithmic variances observed for the 25, 50, 100 and 200-ft data squares (see §2.8) were as follows:

Table 6

Squares	No.	Log variance	$f_{50} - f_{200}$
25-ft squares	11 752	0,623	—
50 ,, ,,	4 186	0,435	0,702
100 ,, ,,	1 263	0,290	0,829
200 ,, ,,	394	0,214	0,801

3.3.2 Semivariogram

The semivariogram analysis was carried out on the cm g/t values, as normalised, for the 50-ft blocks using 8 directional segments and the lag categories as shown in Fig. 37. The total number of pairs processed exceeded 300 000 up to lags of 1125 ft. The results for lags up to 425 ft showed a good agreement with the straight line model and are summarised in Table 7. A least squares fit (see §2.9.2) of the combined semivariogram values in the last column against the corresponding average lag distances (column 2) indicated the parameters for the combined semivariogram as shown in the table; thus

$N_{50} = 0{,}2042$ and $3a = 0{,}0702$.

The bottom row of figures in the table shows the f values calculated from (68) using these N_{50} and $3a$ values and the average semivariogram and lag distance values for the 8 directional segments as shown in the last two rows but one.
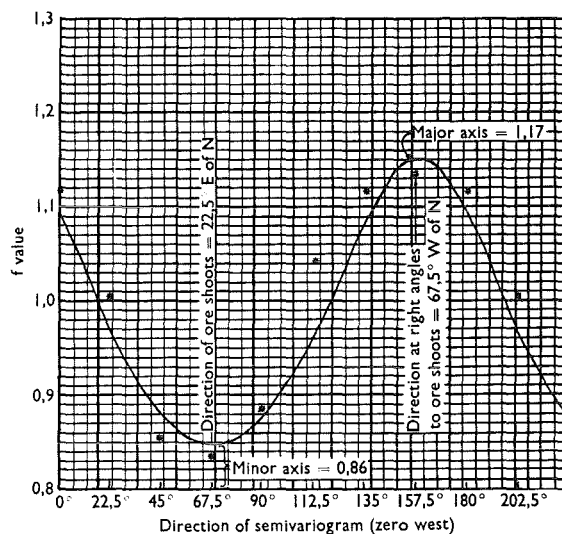

Fig. 38.   Estimation of anisotropy for a section of the Hartebeestfontein mine.

These f values are plotted in Fig. 38 together with the fitted curve indicating:

Direction of major axis (peak of curve) = 157½°
= 67½° W of N
Direction of ore shoots (minor axis) = 22½° E of N
Semi-major axis a = 1,17
Semi-minor axis b = 0,865
Eccentricity of ellipse   = 0,67

Table 7

Semivariogram analysis of gold values for section of Hartebeestfontein mine

Average direction (degrees) (zero west)	0 E–W	22,5	45 NE–SW	67,5	90 N–S	112,5	135 NW–SE	157,5	Averages for lag categories over all segments i.e. combined semivariogram
Segment number	1	2	3	4	5	6	7	8	
Lag no. — Average distance in 50-ft units				(Shoot direction)				(across shoot)	
1 — 1,0	,2712	—	—	—	,2388	—	—	—	,2550
2 — 1,4	—	—	,2802	—	—	—	,2979	—	,2891
3 — 2,0	,3368	—	—	—	,2754	—	—	—	,3061
4 — 2,2	—	,3275	—	,3063	—	,3191	—	,3420	,3237
5 — 2,8	—	—	,3347	—	—	—	,3376	—	,3362
6 — 3,1	,3576	,3432	—	,3160	,3107	,3447	—	,3640	,3394
7 — 4,1	,3805	,3663	,3350	,3371	,3419	,3625	,4006	,3859	,3637
8 — 5,1	,3763	,3762	,3454	,3515	,3727	,3754	,4052	,3854	,3735
9 — 6,1	,4073	,3802	,3604	,3384	,3820	,3918	,3986	,4024	,3826
10 — 7,1	,4146	,4005	,3537	,3750	,3803	,4118	,4112	,4276	,3968
11 — 8,0	,4306	,3964	,3761	,3750	,3971	,4300	,4290	,4330	,4084
Averages for segments	,3719	,3700	,3408	,3428	,3374	,3765	,3829	,3915	$N = 0{,}2042^{*}$
Average distances (geometric)	3,8	4,7	4,3	4,7	3,8	4,7	4,3	4,7	$3a = 0{,}0702^{*}$
f values (see (68))	1,1189	1,0026	,8546	,8377	,8886	1,0415	1,1179	1,1322	$\bar{f} = 0{,}9993$

*See §2.9.2.

These parameters fully define the estimated anisotropic de Wijsian semivariogram — Equation (67) — as follows:

$$\gamma(h) = 0{,}2042 + 0{,}0702 \ \sqrt{\left\{ \frac{1{,}0242}{0{,}748 \cos^2 (\varDelta - 157\tfrac{1}{2}°) + 1{,}369 \sin^2 (\varDelta - 157\tfrac{1}{2}°)} \right\}} \ \log \frac{h}{0{,}446}$$

where h is measured in 50-ft units.

Figure 39 shows the observed semivariograms values — (i) combined, (ii) in the direction of the ore shoots, and (iii) at right angles to the ore shoots as shown in Table 7, last column and columns for segments 8 and 4, respectively — as well as the three fitted straight lines from the above equation up to a lag of 10 units (500 ft). The fits for the other 6 directions are not shown for practical reasons but the overall fit was satisfactory.

3.3.3 Practical checks on formulae

As a practical comparison of (58a) and (58b) the nugget effects were estimated for the 100-ft and 200-ft squares as follows:

$3\alpha = 0{,}0702$; $N_{50} = 0{,}2042$; $f_{100} = 0{,}829$; $f_{200} = 0{,}801$
Adjusted values (-40%); $f_{100} = 0{,}497$; $f_{200} = 0{,}481$
From (58a): $N_{100} = 0{,}11$; $N_{200} = 0{,}07$.
From (58b), which is preferred:

$$N_{100} = \sigma^2_{100} - \{\sigma^2_{50} - N_{50} - (3\alpha/2) \log 4\}$$

$$= 0{,}290 - (0{,}435 - 0{,}2042 - 0{,}486) = 0{,}11$$

and $N_{200} = 0{,}08$.

A practical check of the relationships between semivariograms for different sizes of support and of equations (41), (50) and (58b) was provided by a parallel variogram analysis for this section based on 100-ft square units.

Application of (50) to the 100-ft parameters shows a nugget effect for 50-ft squares of 0,185 compared with the value estimated directly from the 50-ft semivariogram of 0,204; also N_{100} determined directly at 0,0883 compares with 0,11 as calculated from N_{50}. The discrepancies are probably due to
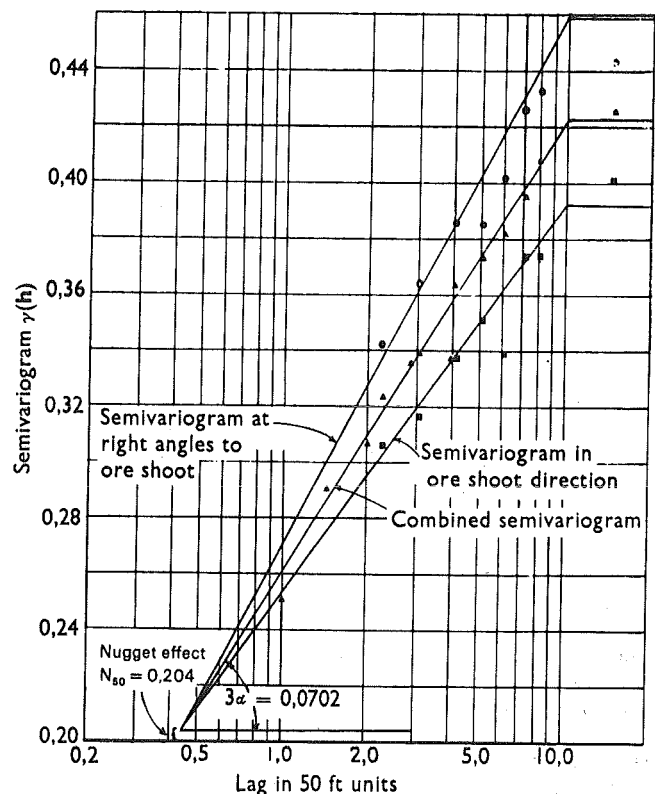

Fig. 39. Anisotropic semivariogram pattern for a section of the Hartebeestfontein mine.

Table 8

Sizes of support areas	50-ft Squares	100-ft Squares
Logarithmic variance (σ^2)	0,4351	0,2900
Logarithmic nugget effect (N)	0,2042	0,0883
Absolute dispersion (α)	0,0234	0,0234
Anisotropy: eccentricity·	0,674	0,674
direction (θ)	$157\frac{1}{2}°$	$146°$

minor deviations from the straight line model for short lags. It should also be borne in mind that in this example the values for 100-ft squares are based on the arithmetic averages of the component 50-ft squares values, and if the log variance of the latter within the former is not constant· the variances and nugget effects for the 100-ft square values will be slightly higher than if these values had been based on the averages of the log transformed values of the 50-ft squares. This factor is usually not significant but could have a small effect on (41), (50) and (58b) as applied above. This is so because, for the lognormal distribution of 50-ft squares within 100-ft squares:

log arithmetic mean = log geometric mean + $\frac{1}{2}$ log variance,

                                         (see (1))

therefore $\log m_{100} = \log g_{100} + \frac{1}{2}\,\sigma^2_{50/100}$

and error variance (log m) = error variance (log g)

                               + $\frac{1}{4}$ error variance of σ^2.

Therefore, $\sigma^2_m = \sigma^2_g + \frac{1}{4}$ error variance of $\sigma^2_{50/100}$

and in the above example:

     $0,2900 = \sigma^2_g + \frac{1}{4}$ error variance of $(0,1451 + 0,2042$

                                     $= 0,3493)$.

Even if the 95% range ($\pm$ 2 standard deviations) for the error distribution of 0,3493 is accepted at, say, zero to 0,6986,

     S.D. = 0,1746, variance = 0,0305, $\frac{1}{4}$ variance = 0,0076,

so that the difference between σ^2_m and σ^2_g will be only some $2\frac{1}{2}$%.

A practical confirmation of (51) — that covariances can be accepted as being independent of the size of the ore units used — is also provided by Table 8:

Cov (**h**) = population variance – semivariogram value.

On 50-ft squares:

    Cov (**h**) = 0,4351 – 0,2042 – (3/2)0,0234 log (**h**/0,446)

             = 0,203 – 0,0351 log **h**.

On 100-ft squares:

    Cov (**h**) = 0,2900 – 0,0883 – (3/2)0,0234 log ($\frac{1}{2}$**h**/0,446)

             = 0,200 – 0,0351 log **h**,

where **h** is measured in 50-ft units.

3.3.4 *Validity of the model for anisotropy*

The validity of the model providing for semivariograms radiating from a common point (representing the nugget effect at a lag of 0,446 units), is difficult to check from the limited evidence available from individual mine sections. However, the semivariogram results for 5 sections of the Hartebeestfontein mine, including the one analysed above and representative of about half of the total area exposed on the Vaal reef horizon in the mine, were combined by pooling the differences between the observed semivariogram values

in the 'shoot' direction and at right angles to it for each section and for each lag category. The results are shown in Table 9.

Table 9

Lag category		Average lag distance (ft)	Difference in $\gamma(h)$ in shoot direction and at right angles thereto
Upper limit in 50-ft units	Feet		
3,5	0/175	112	0,0209
4,5	176/225	156	0,0282
5,5	226/275	204	0,0280
6,5	276/325	256	0,0299
10,5	326/475	350	0,0391

These differences, plotted against the logs of the average lags show a reasonable straight line trend reducing to zero at a lag, of about $0,446 \times 50$ ft = 22,3 ft.

For lags greater than those shown in Table 9 the differences for the mine section concerned appear to stabilise and also correspond to a weak hole-effect for the combined semivariogram. The latter appears to level off over the longer lags up to the maximum lag of interest, which is some 1100 ft. The model accepted in this case was therefore the fitted model up to a maximum lag of 500 ft, at which lag the combined semivariogram value from the model equals the average of the observed values for lags of 500 to 1100 ft. For these longer lags the anisotropic values calculated for a 500 ft lag were therefore accepted. The relevant observed semivariogram values for lags up to 1100 ft and the corresponding model accepted (combined, shoot direction and at right angles to shoot direction) are also shown on Fig. 39. It appears that it would have been even more realistic to have used a model allowing for a reduction in the level of the anisotropy for the longer lags.

3.3.5 *Kriging applications*

3.3.5.1 *The model*

The key parameters as estimated and required for setting up the matrix equations were therefore:

Variances and nugget effects:

    $\sigma^2_{50} = 0,435,$ $N_{50} = 0,2042,$

    $\sigma^2_{100} = 0,290,$ $N_{100} = 0,0883,$

    $\sigma^2_{200} = 0,214,$

    $[\sigma^2_{50}] = \sigma^2_{50} - N_{50} = 0,231,$

    $[\sigma^2_{100}] = \sigma^2_{100} - N_{100} = 0,202.$

Covariances:

    $\sigma_{50/50} = [\sigma^2_{50}] = 0,231$

    $\sigma_{50/50+h} = \sigma_{50/100+h} = \sigma_{50/200+h} = \sigma_{100/100+h}$

          $= \sigma_{100/200+h} = \sigma_{200/200+h}$

          $= \sigma^2_{50} - N_{50} - 3\alpha f \log \dfrac{h}{0,446}$

which, from §3.3.2,

          $= 0,231 - 0,0702\, f \log \dfrac{h}{0,446}$

where

$$f = \left\{ \frac{1,0242}{0,718 \cos^2 (\Delta - 157\tfrac{1}{2}°) + 1,369 \sin^2 (\Delta - 157\tfrac{1}{2}°)} \right\}^{\tfrac{1}{4}}$$

In the above formula **h** is the distance between data block centres expressed in 50-ft units (maximum $= 10$) and Δ is the direction of lag **h** in degrees measured anticlockwise with zero west.

3.3.5.2 Regression or elementary kriging example

Take now as an example a simple case where the central 100-ft square (A) is to be valued from data available only in data square $D1$ (see Fig. 28).

Distance $\quad$ **h** $= \sqrt{300^2 + 300^2} = 424$ ft $= 8,48$ units

direction $\Delta = 225°$

$$\sigma_{A/D1} = 0,231 - 0,0702 \left(\frac{1,0242}{0,1095 + 1,1685} \right)^{\tfrac{1}{4}} \log \frac{8,48}{0,446}$$

$$= 0,231 - 0,0628 \log 19,01345$$

$$= 0,046$$

$$\sigma_{D1}^2 = \sigma_{200}^2 = 0,214$$

$$\sigma_A^2 = [\sigma_{100}^2] = 0,202$$

$$r = \frac{\sigma_{A/D1}}{\sigma_{D1}\sigma_A} = 0,221$$

Hence from (71) and (77) with known mean:

$$\sigma_{D1}^2 [\text{weight } (D1)] = \sigma_{A/D1}$$

$$0,214 \, D1 = 0,046$$

$$D1 = 0,046/0,214 = 0,215.$$

Check from (17): $D1 = r \dfrac{\sigma_y}{\sigma_x} = 0,221 \, (0,972) = 0,215.$

Weight for mean $= 1 - D1 = 0,785.$

Kriging error $= \sigma_e^2 = [\sigma_{100}^2] - D1 \, (\sigma_{A/D1})$
(transformed basis)

$$= 0,202 - 0,215 \, (0,046) = 0,192,$$

Also from (82) with mean ignored:

Weight for $D1 = 1.$

$$\sigma_{\text{error}}^2 = 0,202 + 1(0,214) - 2(0,046)$$

$$= 0,324.$$

The mean, if accepted as the block value, will have an error variance of $\quad [\sigma_{100}^2] = 0,202.$

These three error variances demonstrate clearly the importance of using the mean (if available) when valuing an ore block on scanty information.

It is interesting to follow up on the comments made in §2.12.2 regarding the size of the population area to be used. If in this case the true mean of a population area containing the 100-ft block to be valued and measuring, say, 1100 ft × 1100 ft had been known, the variances and covariance used

above would all have decreased by some 0,013 (see Fig. 39), the $D1$ weight would have increased slightly to 0,229, but the kriging error would have decreased significantly from 0,192 to 0,178. The decrease will be far more significant in cases where a 'hole-effect' (such as occurs in this case) is not present over the critical lag intervals, and this confirms that the best policy is to reduce the population areas (or data areas used without mean) to the minimum practical level.

3.3.5.3 Standard weighting tables

The development raises and stope panels in this mine section are parallel to the major dip in a northwesterly-southeasterly direction. In valuing a standard 100-ft square ore block the data available from development and stoped-out areas will generally occur in areas to the northeast or southwest of the block beyond a boundary running roughly in a NW-SE direction. On this basis the main data patterns likely to be available for valuation purposes (bearing in mind considerations of symmetry—or mirror images—and the practical limitation of two data peripheries — see §2.12.2) are listed (by reference to Fig. 28) in Table 10 as well as the corres-

Table 10

Available data pattern (Block nos. as per Fig. 28)	Logarithmic error variances (transformed units)		
	Mean known	(weight)	Mean not used
1. Complete	0,044	(0,011)	0,044
2. A1,2,4; B1/4, 10/12	0,059	(0,116)	0,061
3. A1; B1/3, 11/12	0,084	(0,260)	0,096
4. A1; B1/3, 11/12; C2/4, 10/12; D4,10	0,080	(0,133)	0,081
5. B1,2,12; C1/3, 11/12	0,106	(0,209)	0,112
6. B1; C1,2,12; D2,3, 11/12	0,129	(0,250)	0,134
7. C1; D1/3, 11/12	0,157	(0,343)	0,165
8. D1/2, 12	0,173	(0,512)	0,203
9. D1	0,192	(0,786)	0,325

ponding kriging error variances both for the mean known and the mean unknown or ignored. It will be seen that the use of the mean shows little advantage once its weight drops to below 20 %.

The effectiveness of using a standard weighting table rather

Table 11

Data blocks	Weight 1*	Weight 2*	Data blocks	Weight 1*	Weight 2*
A1,A3	61	—	C3,C9	28	16
A2,A4	59	—	C4,C10	10	6
B1,B7	20	16	C5,C11	8	5
B2,B8	34	29	C6,C12	15	9
B3,B9	30	25	D1,D7	13	6
B4,B10	17	14	D2,D8	27	12
B5,B11	20	16	D3,D9	25	9
B6,B12	25	21	D4,D10	6	1
C1,C7	16	10	D5,D11	1	−4
C2,C8	30	18	D6,D12	10	4
		Weight for mean = 60			

*Weight 1 applies if a data block concerned is the first in the relevant direction (see Fig. 28) and weight 2 if it is the second data block available in that direction.

Table 12

Data pattern	(i) Optimum weights	(ii) Weights from Table 11	(iii) Orthodox valuation	(iv) Valuation (iii) with regression
	Logarithmic error variances			
(1) B1/4, 10/12;				
C1/4, 10/12	0,0919	0,0939	—	—
B1/3, 11/12	—	—	0,1197	0,1026
Relative efficiencies	100%	97,9%	76,8%	89,6%
(2) B1/2, 9/12;				
C1/2, 9/12	0,0948	0,0988		
B1/2; 9/12	—	—	0,1223	0,1060
Relative efficiencies	100%	96,0%	77,5%	89,4%

than the optimum weights for each specific data pattern (see §2.8) can be judged by comparing the resultant error variances with those shown in Table 10. The standard weighting table used is shown in Table 11, which was determined on the basis of the relative optimum weights established for the patterns in Table 10. The sum of the relative weights for each pattern, including the weight for the mean, are set at a level inversely proportional to the corresponding log error variance (Krige, 1966a). The weights so obtained for each data square were finally averaged and rounded off.

Table 12 shows in summarised form a comparison (typical for the Hartebeestfontein mine) between the logarithmic error variances (transformed basis) of the grade estimates for a central 100-ft square block (Fig. 28), based on two specified data patterns and using the mean together with:

(i) Optimum weights for 2 peripheral data blocks in each direction where available;

(ii) suboptimum weights based on the weighting table in Table 11 for the same data blocks as in (i);

(iii) equal weights for the first peripheral data blocks only, comparable to an orthodox peripheral valuation; and

(iv) as in (iii) but with a suitable weight for the mean, comparable to an orthodox valuation with regression (see §1.18).

The relative efficiencies shown in Table 12 are typical for the Vaal Reef in the Hartebeestfontein mine and illustrate clearly the improvements gained in the stages of development from orthodox peripheral valuations (iii), to straight regression of these valuations (iv), and to kriging based on a standard weighting table (ii). Furthermore, the further improvement in efficiency to be obtained from using optimum weights rather than the standard weighting table, is very small.

On other mines and reefs, such as in the Loraine gold mine in the Orange Free State, where the nugget effects are substantially higher, the improvements in relative efficiencies are even more significant than at Hartebeestfontein. The reason for not using optimum weights is still mainly one of economics, the cost ratio of procedure (i) compared with (ii) being of the order of 8 to 1 or some R1600 as against R200* for a complete grid covering the whole mine. As is evident from Table 12, the sacrifice in efficiency in not using the optimum procedure is negligible.

3.3.6 Gains from improved procedures for a low-grade property

If the progressive improvements in techniques as demon-

strated in Table 12 have no significant practical effect on a mine's profitability, the advantages to be gained would be purely academic. However, it is interesting to examine the position for a low-grade property showing typical levels of improvements — somewhat higher than in Table 12 — with a potential of, say, 50 million tons of unmined ore and the following other characteristics (1977 financial figures):

Average recovery grade with no selective mining . . . . . . 5,4 g/t.
Operating costs per ton milled . . . . R25.
Breakeven recovery with gold at $150/oz (R4193/kg). . 6,0 g/t.

Effective logarithmic variances	*Value*	*Symbol* (see §1.20)
Actual block values	0,15	σ_y^2
Actual peripheral values	0,15	σ_p^2
Observed peripheral values	0,25	$\sigma_x^2 = \sigma_p^2 + \sigma_e^2$
Regressed block values	0,073	$\sigma_z^2 = \sigma_y^2 - \sigma_{ze}^2$
Kriged block values	0,09	$\sigma_z^2{}^* = \sigma_y^2 - \sigma_k^2$

Logarithmic error variances

Peripheral values vs. periphery	0,10	σ_e^2
Peripheral values vs. block $(r_1 = 0,9; r = 0,7)$	0,13	σ_{xe}^2
Regressed block values	0,077	σ_{ze}^2
Kriged block values	0,06	σ_k^2

The overall potential profit from the payable ore would then be as follows (See also Krige 1981):

Valuation technique	*Orthodox*	*Regression*	*Kriging*
Block variance	0,25	0,073	0,09
Payable tons (%)	32,5*	30*	31*
millions	16,2	15,0	15,5
Payable grade (g/t)	7,05†	7,18*	7,37*
Total revenue (R million)	479	452	479
Total costs (R million)	405	375	388
Total working profit (R million)	74	77	91
Gain over orthodox procedure (R million)	—	3 (4%)	17 (23%)

*From Fig. 4 for variances as shown and (pay limit/mean)=6/5,4
= 1,11.

†For $r=0,7$ the pay limit of 6 g/t after regression corresponds to an actual value of 5,9 g/t; the average of the actual block values above 5,9 g/t is determined using the variance of the regressed blocks, i.e. 0,073 (see (19b) and §1.18–1.20).

3.3.7 Semivariogram on unweighted versus weighted data

The advisability of fitting a combined semivariogram model to adequate data on an unweighted basis, as referred to in §2.9.2, is demonstrated by the analysis of data from a section of the Loraine mine in the Orange Free State. This section covers one of a series of Elsburg reefs which are economically mineralised only over a narrow belt, some 300 metres (1 000ft) wide for many kilometres along the suboutcrops of these reefs. These Elsburg reefs deteriorate into uneconomic grit bands further away from the suboutcrops. The section concerned is, therefore, elongated in shape and the data cover a rectangle of some 200 m × 1 000 m (600 ft × 3 500 ft); also, the ore shoots are orientated parallel to the longer side of the rectangle.

For lags up to some 350–400 ft the available pairs of values

Table 13

Semivariogram for gold values from a section of the Loraine mine

Average direction degrees (zero west)	0	22,5	45	67,5	90	112,5	135	157,5	Averages for lag categories over all segments	
Segment number	1	2	3	4	5	6	7	8		
Lag no. / Average distance	(No. of pairs)	Across ore shoots				Direction of ore shoots			Un-weighted	Weighted
1 1,00	,2575	—	—	—	,2763	—	—	—	,2669	,2670
2 1,41	—	—	,3591	—	—	—	,3247	—	,3419	,3416
3 2,00	,4552	—	—	—	,4000	—	—	—	,4276	,4267
4 2,24	—	,4805	—	,4985	—	,4062	—	,4583	,4609	,4590
5 2,83	—	—	,5511	—	—	—	,4816	—	,5163	,5139
6 3,11	,5929	,6142	—	,5730	,5050	,4777	—	,5618	,5541	,5500
7 4,08	,6769	,7014	,6840	,6349	,5296	,5045	,5800	,6426	,6192	,6093
8 5,12	,6822	,7741	,7415	,6789	,5804	,5073	,5748	,7184	,6572	,6400
9 6,08	,7842	,8589	,8249	,7623	,6615	,4866	,5932	,7212	,7116	,6777
10 7,06	,8805	,8563	,8386	,7920	,6688	,4778	,6070	,7579	,7349	,6836
11 7,99	,8178	,8249	,7382	,7939	,7077	,5198	,5981	,8260	,7283	,6792
12 (No. of pairs)	(342)	(277)	(400)	(1020)	(2158)	(845)	(1966)	(962)		
9,89	,9327	1,0758	,7535	,7651	,7279	,5670	,6187	,8984	,7924	,6923
13 (No. of pairs)	(112)	(74)	(113)	(520)	(2136)	(3336)	(1392)	(420)		
12,92	1,0699	1,6607	,7720	,6045	,7660	,6009	,6952	,9525	,8902	,6968
14 (No. of pairs)	(26)	(10)	(29)	(207)	(2012)	(3407)	(1116)	(128)		
15,75	,6609	1,7740	,4533	,5141	,7926	,6120	,6785	1,5566	,8802	,6860
15 (No. of pairs)	—	—	(33)	(182)	(2746)	(3858)	(1059)	(111)		
20,03	—	—	,8682	,6341	,7470	,6231	,7081	,9737	,7590	,6677
Averages for segments	,7101	,9621	,6895	,6592	,6136	,5257	,5873	,8243		

are, therefore, more or less evenly spread over the various directions, but for longer lags the direction of available pairs will obviously tend to coincide more and more with that of the longer side of the rectangle covering the mine section. This
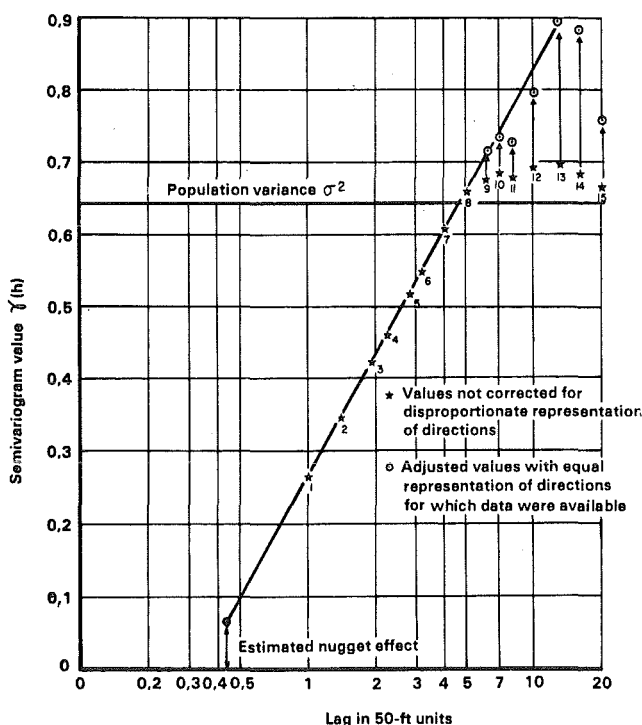

Fig. 40. Combined variogram patterns for a section of the Loraine gold mine showing the bias which can be introduced in weighting variogram values with data frequencies.

direction in turn, being parallel to the ore shoot direction, will correspond to the below average semivariogram values, and for the very long lags the combined semivariogram value on a weighted basis will, therefore, represent virtually only this direction — in other words, it will be artificially depressed. This is demonstrated clearly by the results summarised in Table 13 and Fig. 40. On the weighted basis the combined semivariogram appears to level off at a value just above the population variance — indicating a hole-effect — whereas the unweighted data show the de Wijsian linear trend continuing up to a lag of at least 16 units or 800 ft. It is obvious that if the direction of the ore shoots had been parallel to the short side of the rectangular area, the longer lags would have corresponded to the above average semivariogram values, and the weighted combined variogram would then have shown an upward curvature from the theoretical straight line and could have been mistaken for an exponential model (David, 1977).

Note that on the program now used (1979), lower weights would have been allocated to semivariogram values based on fewer than 200 pairs.

3.4 Examples of geostatistical analyses on the Prieska copper mine

3.4.1 *Contour surfaces for hanging- and foot-wall data* (Krige and Rendu, 1975)

The massive copper and zinc sulphide ore body being exploited at the Prieska mine in the north-eastern Cape Province of South Africa dips steeply and is of highly variable width. Although the limits of the massive sulphides often coincide with the hanging- and foot-walls for mining purposes, this is not general. For mine planning, the positions of these walls have therefore to be determined not only on geological but mainly on economic considerations based on core assays from underground boreholes drilled on an irregular grid. When changes in metal prices and costs of production occur,

the economic limits of the ore body, that is, the hanging- and foot-walls, are affected and this necessitates adjustments in the mine blasting layouts. A poor definition of the hanging- and foot-wall surfaces will obviously result in excessive dilution of the economic ore and/or in undue amounts of economic ore left unmined.

A comprehensive computerised data base has been established covering all survey, geological, structural and analytical data from the underground boreholes. These holes are drilled on an irregular 15 m × 15 m grid, intersect the ore body at various angles, and are seldom normal to the plane of the ore body. In addition data from other ore-exposures in development are recorded (Krige, 1973). This data base is accessed by a suite of ore reserve programs which first provide an updated version of a 'profit formula' and which is applied to the assay-data from each borehole to define the optimum economic positions for the hanging- and foot-walls in terms of three-dimensional coordinates. For both hanging- and foot-walls these positions in three-dimensional space correspond to data points on the two physical surfaces which define the ore body to be mined. These data points, when plotted on a vertical projection, show an irregular pattern of distribution corresponding to extremely variable data densities. For this reason meaningful vertical and horizontal sections cannot be drawn unless these surfaces are first defined.

The fitting of adequately representative hanging- and foot-wall surfaces is of particular importance in the calculation of ore reserve tonnages, for detailed blasthole layouts and for mine planning in general. The data are real and not subject to error; consequently the surfaces have to fit the observed data very closely and one must avoid anomalous estimates between the data points. Serious problems were encountered in the trial application of various standard computer contouring packages and techniques. However, analyses have shown that a simple modified form of universal kriging gives satisfactory results and is comparatively easy and inexpensive to apply.

The three-dimensional coordinates of the hanging- and foot-wall intersections from underground boreholes are converted first to their corresponding values relative to a vertical plane parallel to the regional strike of the ore body. Two of the coordinates then fix the position of a data point on the vertical projection and the third coordinate corresponds to the normal deviation of the data point from the projection plane. This third coordinate value then forms the observed variable for which a contour surface is required relative to the projection plane.

Standard kriging of the observations can satisfy the constraint for exact fitting at the data points but does not accommodate the presence of strong trends and sharp local changes in the data. Marked regional trends are present in the form of regional changes in both strike and dip of the ore body. Universal kriging can overcome the problems of trends, provided a satisfactory regional mathematical trend surface can be fitted to the data, or if the data are first split into subsections of a size which will enable satisfactory trend surfaces to be fitted to these individual subsections.

For the complex type of surface encountered at Prieska, satisfactory trend surfaces could not be fitted. Subdivision was, therefore, pursued to the ultimate but with the simplest
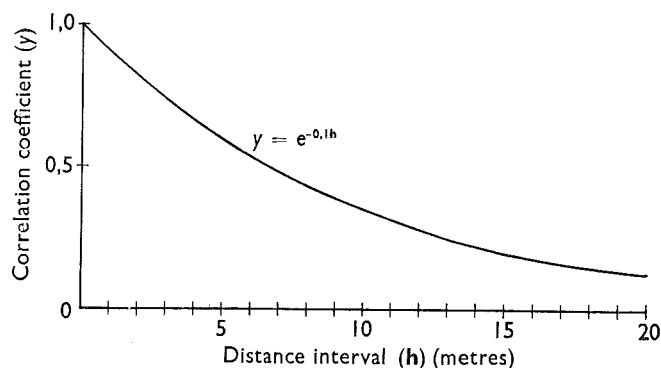

Fig. 41.   Correlogram of residuals for the hanging- and foot-wall surfaces at the Prieska copper mine.

type of surface, i.e. a plane, fitted to a single aureole of data points around each grid point at which a value for the required surface was to be estimated. This modified version of universal kriging was found to be satisfactory in all respects.

The available data were first analysed on the following basis. A 15 m × 15 m grid was superimposed on the data and at each grid point the first aureole of 8 data points was selected by finding the nearest data point in each of 8 segments or octants. A plane surface was then fitted by least squares to the set of data points at each grid point and the corresponding residuals — the data point value less the value of the plane surface — determined. For each pair of residuals within such a set the sum, difference and products, as well as the squares of the residuals, were stored in a range of categories of distances and directions between the two points in the pair. These were pooled over all the grid points covering the ore body and were then used to establish a two-dimensional semivariogram, covariogram or correlogram. The patterns disclosed were reasonably isotropic.

Satisfactory results were obtained using for the correlogram the exponential function $y = e^{-0,1h}$, where y = the correlation coefficient and h = lag interval. Figure 41 is a graphical representation of the correlogram function used. The semivariogram approach would have given the same results. The individual planes were fitted originally by least squares assuming no intercorrelations between residuals, but the appropriateness of the model was also confirmed with the correlogram pattern accepted as known a priori and used in the theoretically correct least squares procedure of fitting each plane.

Having decided on the correlogram function to be used and following the fitting of a plane at each grid point, normal kriging of the residuals was effected and the kriged residual value then added to the value of the fitted plane at the grid point. The resultant grid of estimates of the hanging-wall and foot-wall could then be used directly for volume, tonnage, strike, dip and normal width calculations. For purposes of planning the fans of blast drillholes spaced 1½ metres apart on strike, a close grid of 3 m × 3 m was used for establishing points on the relevant surfaces. All the sections through the ore body, as required for detailed planning, were then calcu-

Table 14

Frequency distribution of Cu% values

Category limits		0,2		0,46		0,77		1,13		1,54		2,02		2,58		3,22		3,97		4,83		Total
Frequency	10		23		30		55		61		80		79		58		25		7		3		431
Cum. freq. %		2,3		7,7		14,6		27,4		41,5		60,1		78,4		91,9		97,7		99,3				100

Table 15

Semivariogram for Cu% values for a section of the Prieska mine

Average direction (degrees zero west)	0	22,5	45	67,5	90	112,5	135	157,5	
Segment number	1	2	3	4	5	6	7	8	
Lag no. Average distance	(across ore shoots)				(shoot direction)				Combined semivariogram
1 1,00	0,035	—	—	—	0,040	—	—	—	0,038
2 1,41	—	—	0,047	—	—	—	0,040	—	0,043
3 2,00	0,050	—	—	—	0,057	—	—	—	0,054
4 2,24	—	0,052	—	0,060	—	0,047	—	0,040	0,051
5 2,83	—	—	0,059	—	—	—	0,055	—	0,057
6 3,11	0,046	0,056	—	0,051	0,054	0,057	—	0,048	0,052
7 4,09	0,049	0,061	0,060	0,051	0,052	0,052	0,054	0,052	0,054
8 5,12	0,058	0,065	0,055	0,057	0,050	0,052	0,052	0,056	0,056
9 6,08	0,060	0,060	0,052	0,052	0,060	0,054	0,047	0,058	0,056
10 7,06	0,065	0,062	0,060	0,060	0,056	0,052	0,047	0,057	0,058
11 8,01	0,072	0,068	0,062	0,059	0,053	0,048	0,053	0,058	0,059
12 10,00	0,078	0,072	0,066	0,063	0,063	0,056	0,058	0,064	0,065
Averages for segments	0,057	0,062	0,058	0,057	0,054	0,052	0,051	0,055	N =0,0349 $3a$=0,0089
Average distances (geometric)	4,2	5,2	4,8	5,2	4,2	5,2	4,8	5,2	
f values	1,110	1,236	1,068	0,997	0,943	0,794	0,738	0,909	$\bar{f}$ =0,9742

lated by computer using a straightline interpolation between these grid points, and finally drawn on an offline plotter. This approach eliminates the serious anomalies encountered in using other surface fitting techniques (Krige and Rendu, 1975).

3.4.2 Example of a semivariogram analysis for copper grades

The following is a brief summary of a semivariogram analysis of the copper grades for a section of the Prieska mine using the logarithmic-de Wijsian model.

The distribution of the grades (% copper) are shown in Table 14 and gave a very good straight line fit on log-probability paper after addition of a constant of 1,5% (see §1.9). The transformed values, i.e. of log (Cu% + 1,5), were then analysed using the same program as for gold (see §3.3.2) and gave the results summarised in Table 15. The unit lag interval used corresponds to the standard grid interval of 15 metres as used in the final kriging operations (see §3.1.2 (iv) and (vi)). The number of pairs analysed per category for lag categories 1–12 combined over all directions varied between 445 and 10 842, but the results were still somewhat irregular because of the limited number of values available in total (431).

A straight line was fitted to the combined semivariogram values (Table 15, last column), and the f values (bottom line) were fitted satisfactorily with the elliptical anisotropic model (see §2.9) with the following results:

$N \doteq 0,0349$ $\quad 3a = 0,0089$
$a = 1,2$ $\quad\quad b = 0,8,$
$\quad\quad$ eccentricity $= 0,75$
direction of ore shoots (minor axis) $= 112\frac{1}{2}°$
$\quad\quad\quad\quad\quad\quad\quad = 22\frac{1}{2}°$ anticlockwise from true dip.

Recent work has clearly confirmed the substantial advantages to be gained from the application of the lognormal – de Wijsian models over models based on untransformed values.

3.4.3 Metal grades versus accumulations

A practical consideration in dealing with Prieska data is that

of using copper grades (percentages) over the selected borehole widths instead of the theoretically correct measures of accumulations as is customary for a two-dimensional problem. This is done for the following practical reasons:

(i) The irregular outlines of the ore body in many localities tend to make estimates of the normal widths of the ore body somewhat academic and very dependent on the actual geometrical procedures followed in estimating these widths; the accumulations — metal grades × these widths—will obviously be subject to the same reservations.

(ii) The correlation between copper grades and normal widths is low for the mine section in question ($r = 0,2$), whereas the copper accumulations are highly correlated with the widths. The physical hanging- and foot-wall surfaces, and hence the widths, are estimated independently of the metal grades and some critical data, such as hanging- and/or foot-wall exposures where the whole width of the ore body cannot be sampled and for which no corresponding metal grades are available, is used in the process. Straight kriging of accumulations and division by the estimated widths to arrive at the metal grade estimates could, therefore, result in serious local biases in localities where such additional width data are used. The kriging of accumulations rather than of metal grades would, therefore, have to be done on the basis of co-kriging with widths (Matheron, 1971) after establishing the relevant cross-variograms so as to avoid the introduction of such biases.

Under these conditions, and in view of the fact that in the valuation of a massive ore body such as at Prieska metal grades rather than accumulations are the main goal, there is no significant advantage in using the more complicated co-kriging procedures referred to above, and no valid objection can be raised against the direct kriging of the metal percentages.

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(Numbers refer to paragraphs)